GEORGIA INSTITUTE OF TECHNOLOGY
ENGINEERING EXPERIMENT STATION
ATLANTA, GEORGIA

August 26, 1955

Commander
Air Research and Development Command
Post Office Box 1395
Baltimore 3, Maryland

Attention: RDTRRM

Subject: Mathematical Study of Shells
Technical Status Report No. 1, June 1 -- August 31, 1955
Contract No. AF 18(600)-1459
Expenditure Order No. 3-3751
ARDC Project No. (Task No. 37500)

Gentlemen:

The primary object of the study in progress under this contract is the calculation of exact values or of upper and lower bounds for the influence coefficients of thin elastic shells of circular cylindrical, hemispherical, and conical shape. The influence coefficients for such shells are the parameters appearing in the linear relations between the displacement and change of inclination at the edge of the shell and the force and moment applied to the edge.

As now planned, the study will include the physical configurations listed in the following outline.

I. Right circular cylindrical shell

A. Constant wall thickness

1. Semi-infinite length
2. Finite length

B. Linearly varying wall thickness (finite length)

1. Wall thickness decreasing to zero at the unloaded edge
2. Non-zero wall thickness at each edge (thicker edge loaded)

C. Exponentially decreasing wall thickness (semi-infinite length)

II. Hemispherical shell

A. Constant thickness

B. Thickness decreasing exponentially from the edge

III. Conical shell

A. Constant thickness

B. Thickness decreasing exponentially from the loaded edge

The essential information for parts I.A.1,2, I.B.1, and I.C. is included in a thesis prepared at M.I.T. and referred to in the original contract proposal. However, this information requires revision and reorganization to make it suitable for publication. A part of the analysis for the conical shell of constant thickness is available in the literature. The remaining parts of the outline represent new analyses.

During the period covered by this progress report (June 1 - August 31, 1955), the following work has been done.

I.A.2. Cylindrical shell; constant thickness; finite length

The results have been extended beyond those in the literature and beyond those in the thesis previously mentioned. So far as the present study is concerned, they are now complete.

I.B.2. Cylindrical shell; linearly varying thickness; non-zero thickness at each edge

The analysis and the computations have been completed.

II.A,B. Hemispherical shell

The critical steps of the analysis (but not all the details and none of the computations) for both constant and exponentially varying thickness have been completed.

In addition, revision and reorganization of the thesis material on the cylindrical shell is in progress.

For the immediate future the following work is planned.

1. Preparation of a manuscript with the tentative title "Bounds on the Influence Coefficients of Semi-Infinite Circular Cylindrical Shells." The intention is to submit this manuscript for publication in a technical journal and for approval for distribution as an OSR Technical Note.

2. Preparation of a manuscript with the tentative title "Circular Cylindrical Shells with Linearly Varying Wall Thickness." The present plan is to submit this material for approval as a Technical Note and to submit a possibly briefer version for publication in a technical journal.

3. Completion of the work on the hemispherical shell.

These three items are expected to occupy all the time available until December 1, 1955, and perhaps somewhat longer.

Yours truly,

M. B. Sledd
Professor, School of Mathematics

Approved:

J. E. Boyd
Assistant Director (Research)

GEORGIA INSTITUTE OF TECHNOLOGY

ENGINEERING EXPERIMENT STATION

ATLANTA, GEORGIA

November 28, 1955

Commander
Air Research and Development Command
Post Office Box 1395
Baltimore 3, Maryland

Attention: RDTRPM

Subject: Mathematical Study of Shells
Technical Status Report No. 2, September 1 -- November 30, 1955
Contract No. AF 18(600)-1459
Expenditure Order No. 3-3751
ARDC Project No. (Task No. 37500)

Gentlemen:

In Technical Status Report No. 1 covering the period June 1 - August 31, 1955, three items were listed as work planned for the immediate future. The first of these items was preparation of a manuscript with the tentative title "Bounds on the Influence Coefficients of Semi-Infinite Circular Cylindrical Shells."

During the interval September 1 - September 20 approximately eighty percent of this manuscript was written in a form essentially ready for typing. Thereafter academic duties brought the work to a complete stop, and no further substantial progress is expected until the Christmas holidays.

The outline of work planned for the coming months remains as indicated in Technical Status Report No. 1.

Yours truly,

M. B. Sledd
Professor, School of Mathematics

Approved:

J. E. Boyd
Associate Director

AFOSR TN-56-575
ASTIA Document No. 110 397

ENGINEERING EXPERIMENT STATION
of the Georgia Institute of Technology
Atlanta, Georgia

Report No. 1 | Project No. A-231

BOUNDS ON INFLUENCE COEFFICIENTS FOR CIRCULAR CYLINDRICAL SHELLS

by

Eric Reissner and M. B. Sledd

DI

December 1, 1956

MATHEMATICS DIVISION, AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Contract No. AF18(600)-1459 | File Number 2.7
Expenditure Order No. 3-3751

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Bounds on Influence Coefficients for Circular Cylindrical Shells*

by

Eric Reissner and M. B. Sledd

1. Introduction. We consider in the following a problem involving rotationally symmetric deformations of a thin elastic circular cylindrical shell of variable wall thickness and of finite or semi-infinite axial length. One of the two edges of the shell is a free edge ; the other is acted upon by a uniformly distributed bending moment M_o and radial force H_o. The shell deforms under the influence of the moment and force and, in particular, the edge along which M_o and H_o are applied undergoes a radial displacement u_o and a rotation β_o (Fig. 1). Assuming that relations between stresses and strains are linear, we have that the statical quantities M_o and H_o and the geometrical quantities β_o and u_o are related linearly. We may write

$$\begin{aligned} u_o &= - c_{uH} H_o + c_{uM} M_o \\ \beta_o &= \quad c_{\beta H} H_o - c_{\beta M} M_o \end{aligned} \qquad (1.1)$$

or

$$\begin{aligned} H_o &= - k_{uH} u_o - k_{\beta H} \beta_o \\ M_o &= - k_{uM} u_o - k_{\beta M} \beta_o \end{aligned} \quad . \qquad (1.2)$$

The coefficients c will in the following be called <u>direct influence coefficients</u> and the coefficients k will be called <u>inverse influence coefficients</u>. The choice of signs is such that the quantities c and k are positive.

* Supported by the Office of Naval Research under a contract with the Massachusetts Institute of Technology, by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under contract No. AF18(600)-1459, and by the Engineering Experiment Station of the Georgia Institute of Technology. The results for the case of semi-infinite shells represent a chapter of the doctoral dissertation of the second author, accepted by MIT in September, 1954.

Exact determination of the influence coefficients c and k in closed form is possible for shells with constant wall thickness, for shells with linearly varying wall thickness, and for shells with quadratically varying wall thickness. In the present paper we are not concerned with the determination of exact values for the coefficients c and k but rather with the determination of upper and lower bounds for the values of these coefficients. We obtain such bounds by application of the minimum principles for displacements and for stresses of the theory of elasticity.

In comparison with existing earlier work on bounds for influence coefficients in structural mechanics - for instance, for the torsional rigidity constant according to the St. Venant torsion theory - our work in the following is characterized by the fact that we undertake the bounding of several such coefficients simultaneously. This feature brings with it considerations which do not arise so long as only one influence coefficient enters a given problem. Extension of the present work to other problems for beams, plates, and shells, in which arrays of influence coefficients need to be considered, will be seen to be possible.

As an example of the application of our inequalitites for direct and inverse influence coefficients, we consider in some detail the case of a semi-infinite circular cylindrical shell with exponentially varying wall thickness. We note that our results extend to ranges of thickness variations which occur over axial distances too short to permit application of asymptotic integration methods for the solution of the differential equations of the problem.

2. Differential equations and boundary conditions. We designate the radius of the middle surface of the cylindrical shell by a and the axial coordinate by z. Assuming rotationally symmetric stresses and deformations in the shell, we have that the state of stress is represented by stress resultants N_z, N_θ, and H and by stress couples M_z and M_θ, in accordance with Figure 2.

In the absence of distributed surface loads and axial edge loads the axial stress resultant N_z vanishes, and the remaining stress resultants and couples satisfy the following two equilibrium equations.

$$\frac{dH}{dz} - \frac{N_\theta}{a} = 0 , \quad \frac{dM_z}{dz} + H = 0 \tag{2.1}$$

Further relations involving the four quantitites N_θ , H, M_z , and M_θ follow from a consideration of the deformation of the shell element. Let u be the radial deflection of the middle surface of the shell and β the rotation of the middle-surface meridian due to deformation, as indicated in Figure 3. The deflections u and β give rise to a circumferential middle-surface strain ε_θ and to a meridional curvature K_z of the following form.

$$\varepsilon_\theta = \frac{u}{a} , \quad K_z = \frac{d\beta}{dz} \tag{2.2}$$

Assuming isotropy, Hooke's law, and the negligibility of transverse shear deformation, we have the following two stress-strain relations :

$$\varepsilon_\theta = \frac{N_\theta}{C} , \quad K_z = \frac{M_z}{D} . \tag{2.3}$$

In these relations $C = Eh$ and $D = Eh^3/12(1-\nu^2)$, where h is the thickness of the shell wall and E and ν are Young's modulus and Poisson's ratio of the material. Because of the absence of any curvature change K_θ, we have further that the moment M_θ is proportional to the moment M_z, as follows.

$$M_\theta = \nu M_z \tag{2.4a}$$

Because of the absence of an axial stress resultant N_z, the axial middle-surface strain ε_z is given by

$$\varepsilon_z = -\frac{\nu N_\theta}{C} . \tag{2.4b}$$

Introduction of (2.2) into (2.3) leaves us in (2.1) and (2.3) with four equations for the five quantities H, N_θ, M_z, u , and β. A fifth equation is an equation of compatibility of the form

$$\beta = \frac{du}{dz} . \tag{2.5}$$

Of these five equations four are differential equations of the first order and one, the equation involving ε_θ, is of the zeroth order. This means that the system is equivalent to one single fourth order equation and that it requires the statement of four boundary conditions.

Boundary conditions at a free edge are the vanishing of H and M_z at this edge. Accordingly, if the coordinate of the free edge is denoted by z_1, we have as two of a total of four boundary conditions

$$z = z_1 : \quad H = 0 , \quad M_z = 0 . \tag{2.6}$$

We may designate the coordinate of the loaded edge by z_o and the force and the moment acting on it by H_o and M_o. If we prescribe H_o and M_o, we state the remaining two boundary conditions in the form

$$z = z_o : \quad H = H_o , \quad M_z = M_o . \tag{2.7a}$$

Alternately, we may prescribe at the loaded edge the linear and angular displacements rather than the forces and moments causing these displacements. If this is done, equations (2.7a) are replaced by the following two boundary conditions.

$$z = z_o : \quad u = u_o , \quad \beta = \beta_o \tag{2.7b}$$

Our principal interest in what follows consists in the relations (1.1) and 1.2) which connect the quantities H_o and M_o in (2.7a) for the actual solution of the problem with the quantities u_o and β_o in (2.7b) for the same actual solution.

3. Explicit solution for semi-infinite shell of constant thickness. We shall subsequently make use of the known explicit solution of (2.1) to (2.7) for the special case

$$h = h_r = \text{constant}. \tag{3.1}$$

We choose $z_o = 0$, $z_1 = \infty$; and we use the symbol h_r to indicate that we are dealing with a reference thickness. We further introduce a dimensionless coordinate ξ defined by

$$z = a\xi , \tag{3.2}$$

and in the remainder of this section we indicate differentiation with respect to ξ by primes. We then have from (2.2) and (2.3), writing $D = D_r$,

$$M_z = \frac{D_r}{a} \beta' \tag{3.3}$$

and from (2.1)

$$H = -\frac{D_r}{a^2} \beta'' , \quad N_\Theta = -\frac{D_r}{a^2} \beta''' . \tag{3.4}$$

Again from (2.2), (2.3), and (2.1), with $C = C_r$,

$$u = -\frac{D_r}{aC_r} \beta''' . \tag{3.5}$$

Introduction of (3.5) into (2.5) leads to a differential equation for β , as follows:

$$\beta^{IV} + 4\lambda_r^4 \beta = 0 , \tag{3.6}$$

where

$$\lambda_r = \sqrt[4]{\frac{a^2 C_r}{4D_r}} = \sqrt[4]{3(1-\nu^2)} \sqrt{\frac{a}{h_r}} . \tag{3.7}$$

The solution of (3.6) which leads to zero values of H and M_z for $z = \infty$ is of the form

$$\beta = (c_1 \cos \lambda_r\xi + c_2 \sin \lambda_r\xi)\, e^{-\lambda_r\xi} . \tag{3.8}$$

The constants of integration c_1 and c_2 are to be determined from the boundary conditions (2.7), applied for $\xi = 0$. We obtain on the basis of (2.7a)

$$\beta = \left[\frac{a^2}{2\, D_r\lambda_r^2} H_o(\cos \lambda_r\xi + \sin \lambda_r\xi) - \frac{a}{D_r\lambda_r} M_o \cos \lambda_r\xi \right] e^{-\lambda_r\xi} \tag{3.9a}$$

and on the basis of (2.7b)

$$\beta = \left[\beta_o \left(\cos\lambda_r\xi - \sin\lambda_r\xi\right) - \frac{2\lambda_r}{a} u_o \sin\lambda_r\xi \right] e^{-\lambda_r\xi} . \tag{3.9b}$$

The values of M_z , H , N_θ , and u follow by introduction of (3.9) into (3.3) to (3.5). We find

$$u = \begin{cases} \left[-\dfrac{a^3}{2D_r\lambda_r^{\,3}} H_o \cos\lambda_r\xi + \dfrac{a^2}{2D_r\lambda_r^{\,2}} M_o \left(\cos\lambda_r\xi - \sin\lambda_r\xi\right) \right] e^{-\lambda_r\xi} ; \\[2ex] \left[u_o \left(\cos\lambda_r\xi + \sin\lambda_r\xi\right) + \dfrac{a}{\lambda_r} \beta_o \sin\lambda_r\xi \right] e^{-\lambda_r\xi} ; \end{cases} \tag{3.10}$$

$$N_\theta = \frac{C_r}{a} u ; \tag{3.11}$$

$$H = \begin{cases} \left[H_o \left(\cos\lambda_r\xi - \sin\lambda_r\xi\right) + \dfrac{2\lambda_r}{a} M_o \sin\lambda_r\xi \right] e^{-\lambda_r\xi} ; \\[2ex] \left[-\dfrac{4D_r\lambda_r^{\,3}}{a^3} u_o \cos\lambda_r\epsilon - \dfrac{2D_r\lambda_r^{\,2}}{a^2} \beta_o \left(\cos\lambda_r\xi + \sin\lambda_r\xi\right) \right] e^{-\lambda_r\xi} ; \end{cases} \tag{3.12}$$

$$M_z = \begin{cases} \left[M_o \left(\cos\lambda_r\xi + \sin\lambda_r\xi\right) - \dfrac{a}{\lambda_r} H_o \sin\lambda_r\xi \right] e^{-\lambda_r\xi} ; \\[2ex] \left[-\dfrac{2D_r\lambda_r}{a} \beta_o \cos\lambda_r\xi - \dfrac{2D_r\lambda_r^{\,2}}{a^2} u_o \left(\cos\lambda_r\xi - \sin\lambda_r\xi\right) \right] e^{-\lambda_r\xi} . \end{cases} \tag{3.13}$$

Equations (3.9) and (3.10) lead to the following relations between edge displacements and edge forces and couples.

$$\beta(0) = \beta_o = c_{\beta Hr} H_o - c_{\beta Mr} M_o \quad ,$$
$$u(0) = u_o = -c_{uHr} H_o + c_{uMR} M_o \quad , \tag{3.14}$$

where

$$c_{\beta Hr} = c_{uMr} = \frac{a^2}{2D_r\lambda_r^{\ 2}} \ , \quad c_{\beta Mr} = \frac{a}{D_r\lambda_r} \ , \quad c_{uHr} = \frac{a^3}{2D_r\lambda_r^{\ 3}} \tag{3.15}$$

are expressions for the direct influence coefficients for the semi-infinite shell of constant thickness h_r.

Equations (3.12) and (3.13) lead to analogous relations involving inverse influence coefficients, as follows.

$$H(0) = H_o = -k_{uHr} u_o - k_{\beta Hr} \beta_o \quad ,$$
$$M_z(0) = M_o = -k_{uMr} u_o - k_{\beta Mr} \beta_o \quad , \tag{3.16}$$

where

$$k_{uHr} = \frac{4D_r\lambda_r^{\ 3}}{a^3} \ , \quad k_{uMr} = k_{\beta Hr} = \frac{2D_r\lambda_r^{\ 2}}{a^2} \ , \quad k_{\beta Mr} = \frac{2D_r\lambda_r}{a} \tag{3.17}$$

are the values of the inverse influence coefficients for the problem under consideration.

We note that direct and inverse influence coefficients are related as follows:

$$k_{uH} = \frac{c_{\beta M}}{\Delta} \ , \quad k_{uM} = k_{\beta H} = \frac{c_{uM}}{\Delta} \ , \quad k_{\beta M} = \frac{c_{uH}}{\Delta} \ , \tag{3.18}$$

where

$$\Delta = c_{uH}\, c_{\beta M} - c_{uM}^{\ 2} \quad . \tag{3.19}$$

In what follows, the above results, as expressed by (3.14) to (3.17), are to be extended so as to apply to finite shells of variable thickness, in the sense that upper and lower bounds are to be obtained for the values of the coefficients c and k (without the subscript r) which apply when h_r is replaced by some non-constant function $h(\xi)$.

4. Minimum principles for the given shell problem. We may state the two basic minimum principles of the theory of elasticity in the following suitable forms for our present purposes.

(1) Minimum principle for displacements. Among all continuous states of displacements which satisfy all displacement boundary conditions, the actual state (which also satisfies the stress boundary conditions and, through the stress-strain relations, the equilibrium differential equations) is characterized by the condition

$$\frac{1}{2}\int_{z_o}^{z_1} (C\,\varepsilon_\theta^{\,2} + D\,\mathcal{K}_z^{\,2})\,dz + [\,u(z_o)H_o + \beta(z_o)M_o\,] = \text{Min.} \qquad (4.1a)$$

or

$$\frac{1}{2}\int_{z_o}^{z_1} (C\,\varepsilon_\theta^{\,2} + D\,\mathcal{K}_z^{\,2})\,dz = \text{Min.} \qquad (4.1b)$$

Equation (4.1a) is associated with the stress boundary conditions (2.7a) while equation (4.1b) is associated with the displacement boundary conditions (2.7b). Here and subsequently it is understood that $z_1 > z_o$.

(2) Minimum principle for stresses. Among all states of stress which satisfy the equilibrium differential equations and the stress boundary conditions, the actual state (which also satisfies the displacement boundary conditions and, through the stress-displacement relations, the compatibility equation) is characterized by the condition

$$\frac{1}{2}\int_{z_o}^{z_1}\left(\frac{N_\theta^{\ 2}}{C}+\frac{M_z^{\ 2}}{D}\right)dz = \text{Min.} \tag{4.2a}$$

or

$$\frac{1}{2}\int_{z_o}^{z_1}\left(\frac{N_\theta^{\ 2}}{C}+\frac{M_z^{\ 2}}{D}\right)dz + \left[\,H(z_o)u_o + M_z(z_o)\beta_o\,\right] = \text{Min.} \tag{4.2b}$$

Equation (4.2a) is associated with the stress boundary conditions (2.7a) while equation (4.2b) is associated with the displacement boundary conditions (2.7b).

The validity of the above statements may be verified as follows. We consider in addition to the variables ε_θ, K_z, etc., which correspond to the actual state of the system, variations $\delta\varepsilon_\theta$, δK_z, etc., of these variables and comparison functions $\varepsilon_\theta + \delta\varepsilon_\theta = \tilde{\varepsilon}_\theta$, $K_z + \delta K_z = \tilde{K}_z$, etc. We then evaluate the functionals

$$\Pi_{da}(\tilde{u}, \tilde{\beta}) = \frac{1}{2}\int_{z_o}^{z_1}\left(C\,\tilde{\varepsilon}_\theta^{\ 2} + D\tilde{K}_z^{\ 2}\right)dz + \left[\,\tilde{u}\,(z_o)H_o + \tilde{\beta}(z_o)M_o\right] \tag{4.3a}$$

$$\Pi_{db}(\tilde{u}, \tilde{\beta}) = \frac{1}{2}\int_{z_o}^{z_1}\left(C\,\tilde{\varepsilon}_\theta^{\ 2} + D\tilde{K}_z^{\ 2}\right)dz \tag{4.3b}$$

$$\Pi_{sa}(\tilde{H}, \tilde{M}_z) = \frac{1}{2}\int_{z_o}^{z_1}\left(\frac{\tilde{N}_\theta^{\ 2}}{C}+\frac{\tilde{M}_z^{\ 2}}{D}\right)dz \tag{4.4a}$$

$$\Pi_{sb}(\tilde{H}, \tilde{M}_z) = \frac{1}{2}\int_{z_o}^{z_1}\left(\frac{\tilde{N}_\theta^{\ 2}}{C}+\frac{\tilde{M}_z^{\ 2}}{D}\right)dz + \left[\,\tilde{H}\,(z_o)u_o + \tilde{M}_z(z_o)\beta_o\right] \tag{4.4b}$$

to the extent of showing the validity of inequalities of the form

$$\Pi_{da}(\tilde{u}, \tilde{\beta}) \geq \Pi_{da}(u, \beta)\ , \quad \text{etc.} \tag{4.5}$$

We limit ourselves here to the actual derivation in one of the four cases under consideration, say the case given by equation (4.3a). We write

$$\Pi_{da}(\tilde{u},\tilde{\beta}) = \frac{1}{2}\int_{z_o}^{z_1} [C\varepsilon_\theta^{\,2} + DK_z^{\,2} + 2C\varepsilon_\theta\delta\varepsilon_\theta + 2DK_z\delta K_z + C(\delta\varepsilon_\theta)^2 + D(\delta K_z)^2]\ dz$$

$$+\left\{[\,u(z_o) + \delta u(z_o)\,]\,H_o \;+\; [\beta(z_o) + \delta\beta(z_o)\,]\,M_o\right\}$$

$$= \Pi_{da}(u,\beta) + \int_{z_o}^{z_1} (C\varepsilon_\theta\delta\varepsilon_\theta + DK_z\delta K_z)\ dz$$

$$+ [\,\delta u(z_o)H_o + \delta\beta(z_o)M_o\,] + \frac{1}{2}\int_{z_o}^{z_1} [C(\delta\varepsilon_\theta)^2 + D(\delta K_z)^2]\ dz\,. \tag{4.6}$$

Since the last integral in (4.6) is never negative, we shall have established the validity of (4.5) if we show that the terms linear in the variations cancel each other. In order to show that such cancellation does in fact occur, we write

$$\int_{z_o}^{z_1} (C\varepsilon_\theta\delta\varepsilon_\theta + DK_z\delta K_z)\ dz$$

$$= \int_{z_o}^{z_1} \left(N_\theta\frac{\delta u}{a} + M_z\frac{d\,\delta\beta}{dz}\right) dz = \int_{z_o}^{z_1}\left(\frac{dH}{dz}\delta u + M_z\frac{d\,\delta\beta}{dz}\right) dz$$

$$= (H\,\delta u + M_z\,\delta\beta)_{z_o}^{z_1} - \int_{z_o}^{z_1}\left(H\frac{d\,\delta u}{dz} + \frac{dM_z}{dz}\,\delta\beta\right) dz$$

$$= -[\,H_o\delta u(z_o) + M_o\delta\beta(z_o)] - \int_{z_o}^{z_1} H\left(\frac{d\,\delta u}{dz} - \delta\beta\right) dz. \tag{4.7}$$

The integral in (4.7) vanishes since $\tilde{u}$ and $\tilde{\beta}$ and therewith δu and $\delta\beta$ satisfy the compatibility equation (2.5). The integrated portion in (4.7) cancels the integrated portion in (4.6), and we are left with the relation

$$\Pi_{da}(\tilde{u},\tilde{\beta}) = \Pi_{da}(u,\beta) + \frac{1}{2}\int_{z_o}^{z_1} [\, C(\delta\varepsilon_\theta)^2 + D(\delta K_z)^2]\, dz \,, \tag{4.8}$$

which shows that the value of $\Pi_{da}(u,\beta)$ is bounded above by the value of $\Pi_{da}(\tilde{u},\tilde{\beta})$.

In order to state the resultant inequalities in a form which is suitable for what follows, we show next that for all functions u, β, N_θ, M_z which satisfy the differential equations (2.1) to (2.5) we have the following relation.

$$I \equiv \int_{z_o}^{z_1} (C\varepsilon_\theta^{\,2} + DK_z^{\,2})\, dz = \int_{z_o}^{z_1} \left(\frac{N_\theta}{C}^{2} + \frac{M_z}{D}^{2} \right) dz$$

$$= [\, u(z)H(z) + \beta(z)M_z(z)\,]_{z_o}^{z_1} \tag{4.9}$$

If the functions u, β, N_θ, M_z are such that furthermore all the boundary conditions (2.6) and (2.7) are satisfied - that is, the functions u, β, N_θ, and M_z are the actual solution functions of the given boundary-value problem - then we have the further reductions

$$I = -[\, u(z_o)H_o + \beta(z_o)M_o\,] \tag{4.10a}$$

and

$$I = -[\, u_oH(z_o) + \beta_oM_z(z_o)] \,, \tag{4.10b}$$

depending on whether (2.7a) or (2.7b) is prescribed.

Introduction of (4.10) into the inequalities of the form (4.5) which follow from (4.3) and (4.4) leaves us with the following set of specific inequalities:

$$\frac{1}{2}\left[\, u(z_o)H_o + \beta(z_o)M_o \,\right] \leq \frac{1}{2}\int_{z_o}^{z_1}\left[\frac{C}{a^2}\tilde{u}^{\,2} + D\left(\frac{d\tilde{\beta}}{dz}\right)^2\right] dz + \tilde{u}\,(z_o)H_o + \tilde{\beta}(z_o)M_o\ ; \tag{4.11a}$$

$$-\frac{1}{2}\left[u_oH(z_o) + \beta_oM_z(z_o)\right] \leq \frac{1}{2}\int_{z_o}^{z_1}\left[\frac{C}{a^2}\tilde{u}^{\,2} + D\left(\frac{d\tilde{\beta}}{dz}\right)^2\right] dz\ ; \tag{4.11b}$$

$$-\frac{1}{2}\left[u(z_o)H_o + \beta(z_o)M_o\,\right] \leq \frac{1}{2}\int_{z_o}^{z_1}\left[\frac{\tilde{N}_\theta^{\,2}}{C} + \frac{\tilde{M}_z^{\,2}}{D}\right] dz\ ; \tag{4.12a}$$

$$\frac{1}{2}\left[u_oH(z_o) + \beta_oM_z(z_o)\,\right] \leq \frac{1}{2}\int_{z_o}^{z_1}\left[\frac{\tilde{N}_\theta^{\,2}}{C} + \frac{\tilde{M}_z^{\,2}}{D}\right] dz + u_o\tilde{H}(z_o) + \beta_o\tilde{M}_z(z_o)\ . \tag{4.12b}$$

Relation (4.11a) holds for all continuous functions $\tilde{u}$ and $\tilde{\beta}$ which satisfy the compatibility equation $\tilde{\beta} = d\,\tilde{u}/dz$ and for which the integral involved exists.

The validity of (4.11b) requires in addition that $\tilde{u}\,(z_o) = u_o$ and $\tilde{\beta}(z_o) = \beta_o$.

Relation (4.12a) holds for all continuous functions $\tilde{N}_\theta$ and $\tilde{M}_z$ which satisfy the equilibrium differential equations (2.1) and the stress boundary conditions (2.6) and (2.7a). This means that we must have that $\tilde{N}_\theta = -a\, d^2\tilde{M}_z/dz^2$ and $\tilde{M}_z(z_1) = \tilde{M}'_z\,(z_1) = 0$, $\tilde{M}_z(z_o) = M_o$, $\tilde{M}'_z\,(z_o) = -H_o$. Here and in sections 5 and 6 a prime indicates differentiation with respect to z.

The validity of (4.12b) requires satisfaction of the same conditions as in (4.12a) except for the boundary conditions for $z = z_o$.

5. Transformation of inequalities. To express the four inequalities (4.11) and (4.12) in terms of the influence coefficients requires the introduction of various pieces of given information. We show the transformation in connection with the inequality (4.11a) in somewhat greater detail than in connection with the other three inequalities.

We know that the comparison functions $\tilde{u}$ and $\tilde{\beta}$ in (4.11a) need to satisfy no other relation than the relation $\tilde{\beta} = d\,\tilde{u}\,/\,dz$. We further know that the comparison functions should depend linearly on the edge force H_o and edge moment M_o. Accordingly, we set

$$\tilde{u} = H_o\, f_{1a}(z) + M_o\, f_{2a}(z) \quad . \tag{5.1}$$

The functions f_{1a} and f_{2a} are arbitrary except for the requirement that $\tilde{u}$ and $\tilde{\beta}$ be continuous and that the integral in (4.11a) exist.

From (5.1) and the compatibility relation $\tilde{\beta} = d\,\tilde{u}\,/\,dz$ it follows that

$$\tilde{\beta} = H_o\, f_{1a}'(z) + M_o\, f_{2a}'(z) \quad . \tag{5.2}$$

Introduction of (5.1) and (5.2) into (4.11a) results in the following relation.

$$\frac{1}{2}[u(z_o)H_o + \beta(z_o)M_o] \leq \frac{1}{2}\int_{z_o}^{z_1} [\frac{C}{a^2}(H_o f_{1a} + M_o f_{2a})^2 + D(H_o f_{1a}'' + M_o f_{2a}'')^2]\, dz$$

$$+ [\, H_o f_{1a}(z_o) + M_o f_{2a}(z_o)\,]\, H_o$$

$$+ [\, H_o f_{1a}'(z_o) + M_o f_{2a}'(z_o)\,]\, M_o \tag{5.3}$$

It is apparent that the right-hand side of (5.3) is a quadratic form in the quantities H_o and M_o. This suggests that the left-hand side be similarly written by introducing into it equations (1.1), which define the direct influence coefficients c. In this way equation (5.3) is transformed as follows.

$$\frac{1}{2}\,[\, - c_{uH}\, H_o^{\,2} + (c_{uM} + c_{\beta H})H_o M_o - c_{\beta M}\, M_o^{\,2}\,]$$

$$\leq \left[\frac{1}{2}\int_{z_o}^{z_1} \left(\frac{C}{a^2} f_{1a}{}^2 + D f_{1a}''^2\right) dz + f_{1a}(z_o) \right] H_o{}^2$$

$$+ \left[\int_{z_o}^{z_1} \left(\frac{C}{a^2} f_{1a}f_{2a} + D f_{1a}'' f_{2a}''\right) dz + f_{2a}(z_o) + f_{1a}'(z_o) \right] H_oM_o$$

$$+ \left[\frac{1}{2}\int_{z_o}^{z_1} \left(\frac{C}{a^2} f_{2a}{}^2 + D f_{2a}''^2\right) dz + f_{2a}'(z_o) \right] M_o{}^2 \qquad (5.4)$$

Equation (5.4) may be written in abbreviated form. We reverse signs, take account of the fact that $c_{\beta H} = c_{uM}$, and write

$$F_{1a}H_o{}^2 - 2F_{12a}H_oM_o + F_{2a}M_o{}^2 \leq c_{uH}H_o{}^2 - 2c_{uM}H_oM_o + c_{\beta M}M_o{}^2 \quad . \qquad (I)$$

The coefficients F_{1a}, F_{12a}, and F_{2a} are seen to be given in the following form.

$$F_{1a} = -2 f_{1a}(z_o) - \int_{z_o}^{z_1} \left(\frac{C}{a^2} f_{1a}{}^2 + D f_{1a}''^2 \right) dz \qquad (5.5)$$

$$F_{2a} = -2 f_{2a}'(z_o) - \int_{z_o}^{z_1} \left(\frac{C}{a^2} f_{2a}{}^2 + D f_{2a}''^2 \right) dz \qquad (5.6)$$

$$F_{12a} = f_{2a}(z_o) + f_{1a}'(z_o) + \int_{z_o}^{z_1} \left(\frac{C}{a^2} f_{1a}f_{2a} + D f_{1a}'' f_{2a}'' \right) dz \qquad (5.7)$$

Equation (I) is one of the four basic inequalities involving the set of influence coefficients c and k.

We now turn to the inequality (4.11b) which is associated with the displacement boundary conditions (2.7b). Here we set

$$\tilde{u} = u_o f_{1b}(z) + \beta_o f_{2b}(z) \quad . \qquad (5.8)$$

In addition to continuity requirements identical with those imposed on the functions f_{1a} and f_{2a}, we must require of f_{1b} and f_{2b} that $\tilde{u}(z_o) = u_o$; that is, we must have

$$f_{1b}(z_o) = 1, \quad f_{2b}(z_o) = 0 . \tag{5.9}$$

From (5.8) follows by differentiation

$$\tilde{\beta} = u_o f_{1b}'(z) + \beta_o f_{2b}'(z) ; \tag{5.10}$$

and since $\tilde{\beta}(z_o) = \beta_o$, we must also have

$$f_{1b}'(z_o) = 0 , \quad f_{2b}'(z_o) = 1 . \tag{5.11}$$

We further express $H(z_o)$ and $M_z(z_o)$ in terms of u_o and β_o by means of equations (1.2). In this manner we obtain the relation

$$\frac{1}{2} [k_{uH} u_o^2 + (k_{\beta H} + k_{uM}) u_o \beta_o + k_{\beta M} \beta_o^2]$$

$$\leq \frac{1}{2} \int_{z_o}^{z_1} [\frac{C}{a^2} (u_o f_{1b} + \beta_o f_{2b})^2 + D(u_o f_{1b}'' + \beta_o f_{2b}'')^2] \, dz . \tag{5.12}$$

Setting $k_{\beta H} = k_{uM}$ and introducing the coefficients

$$F_{1b} = \int_{z_o}^{z_1} [\frac{C}{a^2} f_{1b}^2 + D f_{1b}''^2] \, dz , \tag{5.13}$$

$$F_{2b} = \int_{z_o}^{z_1} [\frac{C}{a^2} f_{2b}^2 + D f_{2b}''^2] \, dz , \tag{5.14}$$

$$F_{12b} = \int_{z_o}^{z_1} [\frac{C}{a^2} f_{1b} f_{2b} + D f_{1b}'' f_{2b}''] \, dz , \tag{5.15}$$

we may write (5.12) in the following final form.

$$k_{uH}u_o^{\,2} + 2\, k_{uM}\, u_o\beta_o + k_{\beta M}\beta_o^{\,2} \leq F_{1b}u_o^{\,2} + 2\, F_{12b}\, u_o\beta_o + F_{2b}\beta_o^{\,2} \qquad \text{(II)}$$

We next consider the inequality (4.12a). We must choose expressions for $\widetilde{N}_\Theta$ and $\widetilde{M}_z$ which satisfy equilibrium in the interior and along the edges of the shell. Beginning with the problem of satisfying boundary conditions, we assume

$$\widetilde{M}_z = M_o\, g_{2a}(z) + H_o\, g_{1a}(z) \quad , \qquad (5.16)$$

where

$$g_{2a}(z_o) = 1, \quad g_{1a}(z_o) = 0, \quad g_{2a}(z_1) = 0, \quad g_{1a}(z_1) = 0 \; . \qquad (5.17)$$

From (5.16) and (2.1) follows

$$\widetilde{H} = - M_o\, g_{2a}'(z) - H_o\, g_{1a}'(z) \; . \qquad (5.18)$$

Hence, the following further boundary conditions must be satisfied by g_{1a} and g_{2a}.

$$g_{2a}'(z_o) = 0, \; g_{1a}'(z_o) = -1 \; , \quad g_{2a}'(z_1) = 0, \quad g_{1a}'(z_1) = 0 \qquad (5.19)$$

Introduction of (5.18) into the first of equations (2.1) gives $\widetilde{N}_\Theta$ in the form

$$\widetilde{N}_\Theta = -\, a\, M_o\, g_{2a}''(z) - a\, H_o\, g_{1a}''(z) \quad . \qquad (5.20)$$

We now introduce (5.16) and (5.20) into the inequality (4.12a). If we then further express $u(z_o)$ and $\beta(z_o)$ in terms of M_o and H_o by way of (1.1), we again arrive at an inequality between quadratic forms, as follows.

$$c_{uH}H_o^{\,2} - 2\, c_{uM}\, H_oM_o + c_{\beta M}M_o^{\,2} \leq G_{1a}H_o^{\,2} - 2\, G_{12a}\, H_oM_o + G_{2a}M_o^{\,2} \qquad \text{(III)}$$

The quantities G_{1a}, G_{12a}, G_{2a} are given by

$$G_{1a} = \int_{z_o}^{z_1} \left[\frac{a^2}{C} g_{1a}''^2 + \frac{1}{D} g_{1a}^2 \right] dz \quad ; \tag{5.21}$$

$$G_{2a} = \int_{z_o}^{z_1} \left[\frac{a^2}{C} g_{2a}''^2 + \frac{1}{D} g_{2a}^2 \right] dz \quad ; \tag{5.22}$$

$$G_{12a} = -\int_{z_o}^{z_1} \left[\frac{a^2}{C} g_{1a}'' \; g_{2a}'' + \frac{1}{D} g_{1a} \, g_{2a} \right] dz \quad . \tag{5.23}$$

We finally consider the inequality (4.12b), which involves displacement boundary conditions for $z = z_o$. We assume

$$\tilde{M}_z = u_o \, g_{1b}(z) + \beta_o \, g_{2b}(z) \quad . \tag{5.24}$$

The equilibrium equations (2.1) then give

$$\tilde{H} = - u_o \, g_{1b}' (z) - \beta_o g_{2b}' (z) \tag{5.25}$$

and

$$\tilde{N}_\theta = - a \, u_o \, g_{1b}'' (z) - a\beta_o \, g_{2b}'' (z) \quad . \tag{5.26}$$

The stress boundary conditions for $z = z_1$ require that the functions g_{1b} and g_{2b} satisfy the following conditions:

$$g_{1b}(z_1) = g_{2b}(z_1) = g_{1b}' (z_1) = g_{2b}' (z_1) = 0 \quad . \tag{5.27}$$

We now introduce (5.24) to (5.26) into (4.12b) and on the left of (4.12b) introduce the inverse influence coefficients k through equations (1.2). In this way we obtain the inequality

$$G_{1b}u_o^2 + 2\, G_{12b}\, u_o\beta_o + G_{2b}\beta_o^2 \leq k_{uH}u_o^2 + 2\, k_{uM}\, u_o\beta_o + k_{\beta M}\beta_o^2 \quad , \tag{IV}$$

where the coefficients G_{1b}, G_{12b}, G_{2b} are of the following form.

$$G_{1b} = 2\, g_{1b}'(z_o) - \int_{z_o}^{z_1} \left[\frac{a^2 g_{1b}''^2}{C} + \frac{g_{1b}^2}{D} \right] dz \tag{5.28}$$

$$G_{2b} = -2\, g_{2b}(z_o) - \int_{z_o}^{z_1} \left[\frac{a^2 g_{2b}''^2}{C} + \frac{g_{2b}^2}{D} \right] dz \tag{5.29}$$

$$G_{12b} = g_{2b}'(z_o) - g_{1b}(z_o) - \int_{z_o}^{z_1} \left[\frac{a^2 g_{1b}'' g_{2b}''}{C} + \frac{g_{1b} g_{2b}}{D} \right] dz \tag{5.30}$$

6. <u>Bounds for influence coefficients</u>. It is apparent that the inequalities (I) to (IV) may be used to establish inequalities for the coefficients c and k which occur in them.

Equations (I) and (III) allow the following conclusions concerning the direct influence coefficients.

$$F_{1a} \leq c_{uH} \leq G_{1a} \tag{V}$$

$$F_{2a} \leq c_{\beta M} \leq G_{2a} \tag{VI}$$

$$F_{12a} - \sqrt{(c_{\beta M} - F_{2a})(c_{uH} - F_{1a})} \leq c_{uM} \leq F_{12a} + \sqrt{(c_{\beta M} - F_{2a})(c_{uH} - F_{1a})} \tag{6.1}$$

$$G_{12a} - \sqrt{(G_{2a} - c_{\beta M})(G_{1a} - c_{uH})} \leq c_{uM} \leq G_{12a} + \sqrt{(G_{2a} - c_{\beta M})(G_{1a} - c_{uH})} \tag{6.2}$$

Equations (6.1) and (6.2) become usable inequalities if in them we replace the quantities $c_{\beta M}$ and c_{uH} by suitable upper and lower bounds. In this way we obtain

$$F_{12a} - \sqrt{(G_{2a} - F_{2a})(G_{1a} - F_{1a})} \leq c_{uM} \leq F_{12a} + \sqrt{(G_{2a} - F_{2a})(G_{1a} - F_{1a})} \tag{VII}$$

and

$$G_{12a} - \sqrt{(G_{2a} - F_{2a})(G_{1a} - F_{1a})} \leq c_{uM} \leq G_{12a} + \sqrt{(G_{2a} - F_{2a})(G_{1a} - F_{1a})} \tag{VIII}$$

We next consider equations (II) and (IV). These allow the following conclusions concerning the inverse influence coefficients k.

$$G_{1b} \leq k_{uH} \leq F_{1b} \tag{IX}$$

$$G_{2b} \leq k_{\beta M} \leq F_{2b} \tag{X}$$

$$F_{12b} - \sqrt{(k_{uH} - F_{1b})(k_{\beta M} - F_{2b})} \leq k_{uM} \leq F_{12b} + \sqrt{(k_{uH} - F_{1b})(k_{\beta M} - F_{2b})} \tag{6.3}$$

$$G_{12b} - \sqrt{(G_{1b} - k_{uH})(G_{2b} - k_{\beta M})} \leq k_{uM} \leq G_{12b} + \sqrt{(G_{1b} - k_{uH})(G_{2b} - k_{\beta M})} \tag{6.4}$$

Introduction of (IX) and (X) into (6.2) and (6.4) finally gives

$$F_{12b} - \sqrt{(G_{1b} - F_{1b})(G_{2b} - F_{2b})} \leq k_{uM} \leq F_{12b} + \sqrt{(G_{1b} - F_{1b})(G_{2b} - F_{2b})} \tag{XI}$$

and

$$G_{12b} - \sqrt{(G_{1b} - F_{1b})(G_{2b} - F_{2b})} \leq k_{uM} \leq G_{12b} + \sqrt{(G_{1b} - F_{1b})(G_{2b} - F_{2b})} \quad . \tag{XII}$$

7. Influence coefficients for semi-infinite shell with exponentially varying thickness. As an example of application of the foregoing results we consider a shell with thickness variation given in the form

$$h = h_r e^{-\rho z/a} = h_r e^{-\rho\xi} , \tag{7.1}$$

the shell extending from $z_o = 0$ to $z_1 = \infty$. We then have

$$D = D_r e^{-3\rho\xi} , \quad C = C_r e^{-\rho\xi} ; \tag{7.2}$$

and the coefficients F and G assume the following form in accordance with (5.5) to (5.7).

$$F_{1a} = -2 f_{1a}(0) - \int_0^\infty \left(\frac{C_r}{a} e^{-\rho\xi} f_{1a}^{\ 2} + \frac{D_r}{a^3} e^{-3\rho\xi} f_{1a}''^{\,2} \right) d\xi \tag{7.3}$$

$$F_{2a} = -\frac{2}{a} f_{2a}'(0) - \int_0^\infty \left(\frac{C_r}{a} e^{-\rho\xi} f_{2a}^{\ 2} + \frac{D_r}{a^3} e^{-3\rho\xi} f_{2a}''^{\,2} \right) d\xi \tag{7.4}$$

$$F_{12a} = f_{2a}(0) + \frac{1}{a} f_{1a}'(0) + \int_0^\infty \left(\frac{C_r}{a} e^{-\rho\xi} f_{1a} f_{2a} + \frac{D_r}{a^3} e^{-3\rho\xi} f_{1a}'' f_{2a}'' \right) d\xi \tag{7.5}$$

Here f_{1a} and f_{2a} are arbitrary continuously differentiable functions with second derivatives such that the integrals exist, and now and in the following primes indicate differentiation with respect to ξ .

According to (5.13) to (5.15)

$$F_{1b} = \int_0^\infty \left(\frac{C_r}{a} e^{-\rho\xi} f_{1b}{}^2 + \frac{D_r}{a^3} e^{-3\rho\xi} f_{1b}''^2 \right) d\xi \quad , \tag{7.6}$$

$$F_{2b} = \int_0^\infty \left(\frac{C_r}{a} e^{-\rho\xi} f_{2b}{}^2 + \frac{D_r}{a^3} e^{-3\rho\xi} f_{2b}''^2 \right) d\xi \quad , \tag{7.7}$$

$$F_{12b} = \int_0^\infty \left(\frac{C_r}{a} e^{-\rho\xi} f_{1b} f_{2b} + \frac{D_r}{a^3} e^{-3\rho\xi} f_{1b}'' f_{2b}'' \right) d\xi \quad , \tag{7.8}$$

where the continuously differentiable functions f_{1b} and f_{2b} must satisfy the following boundary conditions:

$$f_{1b}(0) = 1, \quad f_{2b}(0) = 0, \quad f_{1b}'(0) = 0, \quad f_{2b}'(0) = a \ . \tag{7.9}$$

According to (5.21) to (5.23)

$$G_{1a} = \int_0^\infty \left(\frac{1}{a\,C_r} e^{\rho\xi} g_{1a}''^2 + \frac{a}{D_r} e^{3\rho\xi} g_{1a}{}^2 \right) d\xi \quad , \tag{7.10}$$

$$G_{2a} = \int_0^\infty \left(\frac{1}{a\,C_r} e^{\rho\xi} g_{2a}''^2 + \frac{a}{D_r} e^{3\rho\xi} g_{2a}{}^2 \right) d\xi \quad , \tag{7.11}$$

$$G_{12a} = -\int_0^\infty \left(\frac{1}{a\,C_r} e^{\rho\xi} g_{1a}'' g_{2a}'' + \frac{a}{D_r} e^{3\rho\xi} g_{1a} g_{2a} \right) d\xi \quad , \tag{7.12}$$

where the functions g_{1a} and g_{2a} must satisfy the following conditions :

$$g_{1a}(0) = 0, \quad g_{1a}'(0) = -a \ , \quad g_{1a}(\infty) = 0, \quad g_{1a}'(\infty) = 0 \ ; \tag{7.13}$$

$$g_{2a}(0) = 1, \quad g_{2a}'(0) = 0 \ , \quad g_{2a}(\infty) = 0, \quad g_{2a}'(\infty) = 0 \ . \tag{7.14}$$

Finally, according to (5.28) to (5.30),

$$G_{1b} = \frac{2}{a} g_{1b}'(0) - \int_0^\infty \left(\frac{1}{a\,C_r} e^{\rho\xi} g_{1b}''^2 + \frac{a}{D_r} e^{3\rho\xi} g_{1b}^2 \right) d\xi \ , \tag{7.15}$$

$$G_{2b} = -2\, g_{2b}(0) - \int_0^\infty \left(\frac{1}{a\,C_r} e^{\rho\xi} g_{2b}''^2 + \frac{a}{D_r} e^{3\rho\xi} g_{2b}^2 \right) d\xi \ , \tag{7.16}$$

$$G_{12b} = \frac{1}{a} g_{2b}'(0) - g_{1b}(0) - \int_0^\infty \left(\frac{1}{a\,C_r} e^{\rho\xi} g_{1b}'' g_{2b}'' + \frac{a}{D_r} e^{3\rho\xi} g_{1b} g_{2b} \right) d\xi \ , \tag{7.17}$$

where g_{1b} and g_{2b} must satisfy the following conditions :

$$g_{1b}(\infty) = g_{2b}(\infty) = g_{1b}'(\infty) = g_{2b}'(\infty) = 0 \ . \tag{7.18}$$

8. Choice of functions f and g and numerical results. In applying the inequalities (I) to (XII) we reduce the percentage difference between upper and lower bounds by suitably determining the functions f and g by means of the direct methods of the calculus of variations.

For the semi-infinite shell with exponentially varying thickness we obtain bounds by taking expressions for the functions f and g which are analogous to the known expressions for the shell with constant thickness.

This means that in equations (7.3) to (7.5) we choose

$$f_{1a} = e^{-\lambda_1\xi} \left(A_1 \cos \lambda_1\xi + B_1 \sin \lambda_1\xi \right) \ , \tag{9.1}$$

$$f_{2a} = e^{-\lambda_2\xi} \left(A_2 \cos \lambda_2\xi + B_2 \sin \lambda_2\xi \right) \ ; \tag{9.2}$$

and we determine the parameters A_n , B_n , and λ_n in such a manner that the two quantities F_{1a} and F_{2a} become as large as possible. With A_n , B_n , and λ_n determined in this manner we then evaluate F_{12a} .

In equations (7.6) to (7.8) we choose

$$f_{1b} = e^{-\lambda_1\xi} (\cos \lambda_1\xi + \sin \lambda_1\xi) \quad , \tag{9.3}$$

$$f_{2b} = \frac{a}{\lambda_2} e^{-\lambda_2\xi} \sin \lambda_2\xi \quad , \tag{9.4}$$

which satisfies the boundary conditions (7.9) ; and we determine λ_1 and λ_2 in such a way that F_{1b} and F_{2b} become as small as possible.

In equations (7.10) to (7.12) we choose

$$g_{1a} = -\frac{a}{\lambda_1} e^{-\lambda_1\xi} \sin \lambda_1\xi \quad , \tag{9.5}$$

$$g_{2a} = e^{-\lambda_2\xi} (\cos \lambda_2\xi + \sin \lambda_2\xi) \quad , \tag{9.6}$$

which satisfies the boundary conditions (7.13) and (7.14). We determine λ_1 and λ_2 in such a manner that G_{1a} and G_{2a} become as small as possible. With λ_1 and λ_2 determined in this manner we then evaluate G_{12a} .

Finally, in equations (7.15) to (7.17) we choose

$$g_{1b} = e^{-\lambda_1\xi} (A_1 \cos \lambda_1\xi + B_1 \sin \lambda_1\xi) \quad , \tag{9.7}$$

$$g_{2b} = e^{-\lambda_2\xi} (A_2 \cos \lambda_2\xi + B_2 \sin \lambda_2\xi) \quad ; \tag{9.8}$$

and we determine A_n , B_n , and λ_n to make G_{1b} and G_{2b} as large as possible.

In this manner we have obtained the results incorporated in Figures 4 to 9. For the details of the calculations we refer to the second-named author's dissertation.

What is of particular interest in regard to the present numerical results is the fact that they apply for thickness variations which occur over axial distances so short that the usual methods of asymptotic integration in thin-shell theory fail to apply.

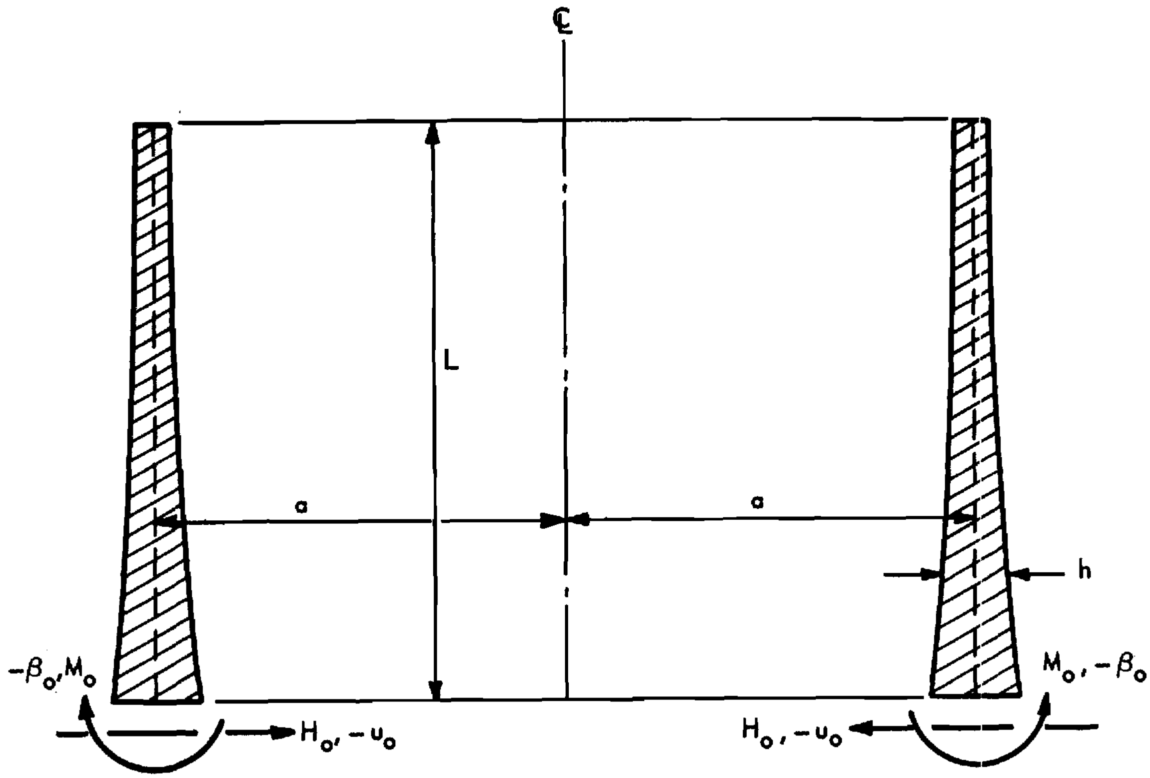


Figure 1. Cross-section of shell loaded along one edge.

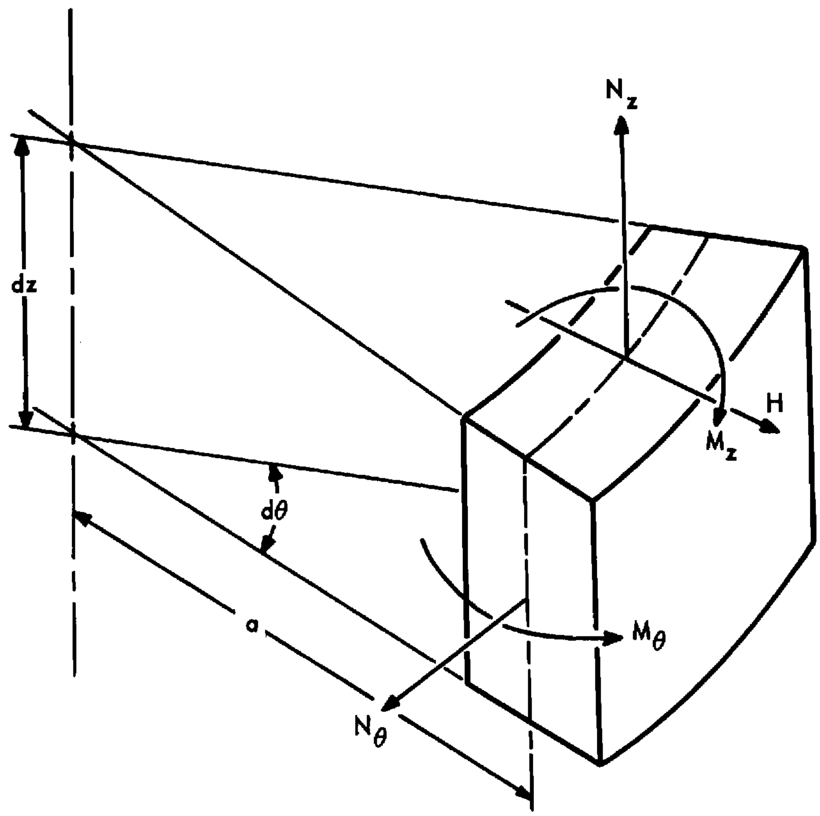


Figure 2. Stress resultants and stress couples acting on an element of the shell.

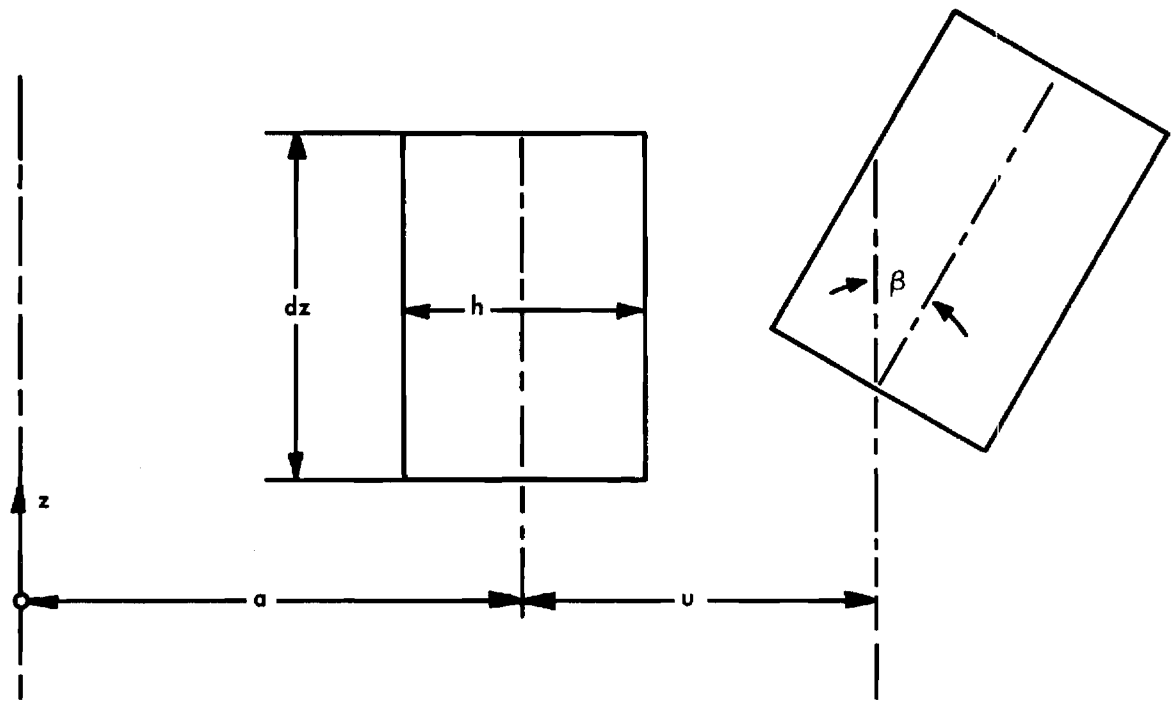
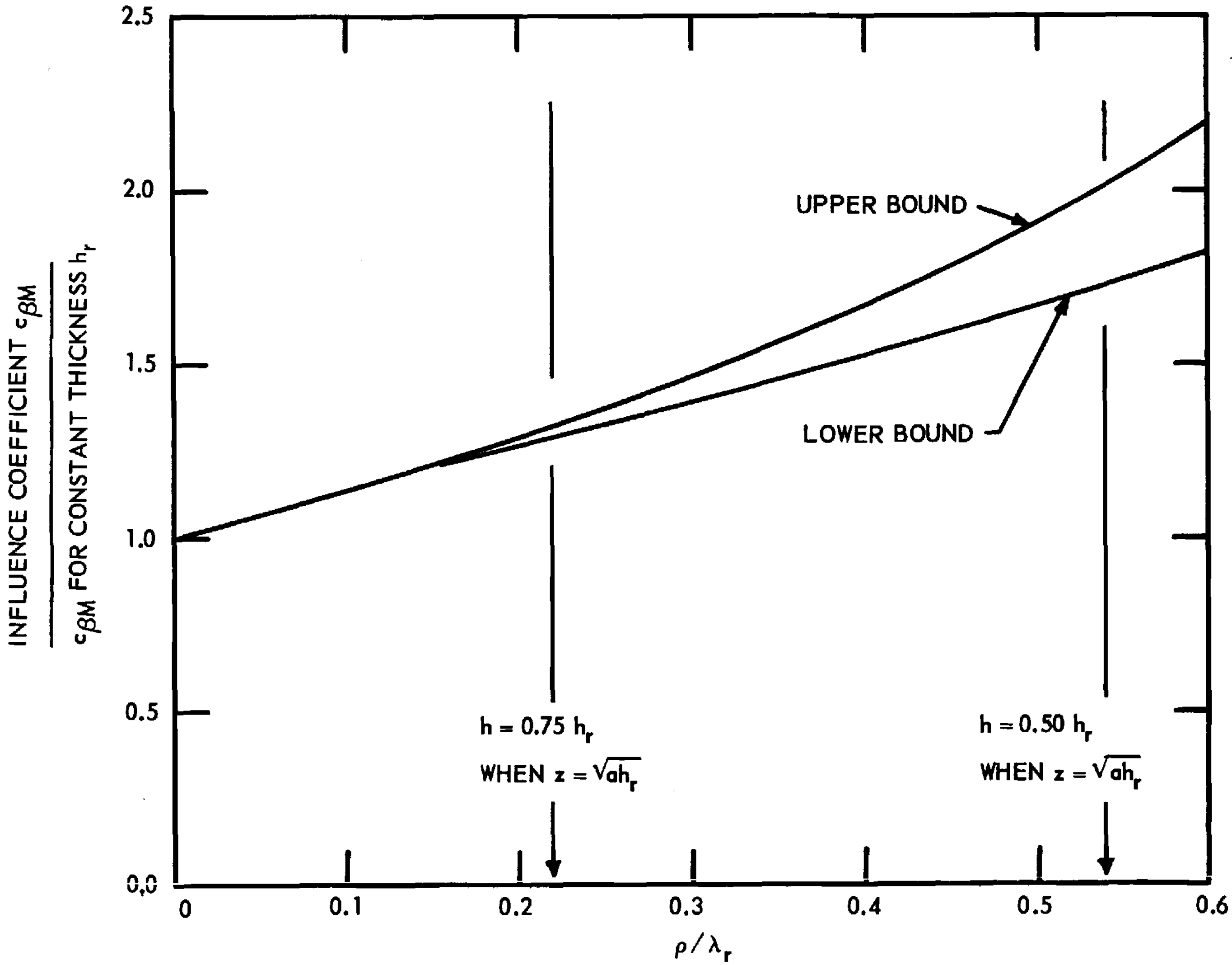


Figure 3. Radial displacement u of a point on the middle surface and rotation β of a meridian through that point.



2.5
2.0
1.5
1.0
0.5
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
INFLUENCE COEFFICIENT $c_{\beta M}$
$c_{\beta M}$ FOR CONSTANT THICKNESS h_r
UPPER BOUND
LOWER BOUND
$h = 0.75\ h_r$
WHEN $z = \sqrt{ah_r}$
$h = 0.50\ h_r$
WHEN $z = \sqrt{ah_r}$
ρ/λ_r

Figure 4

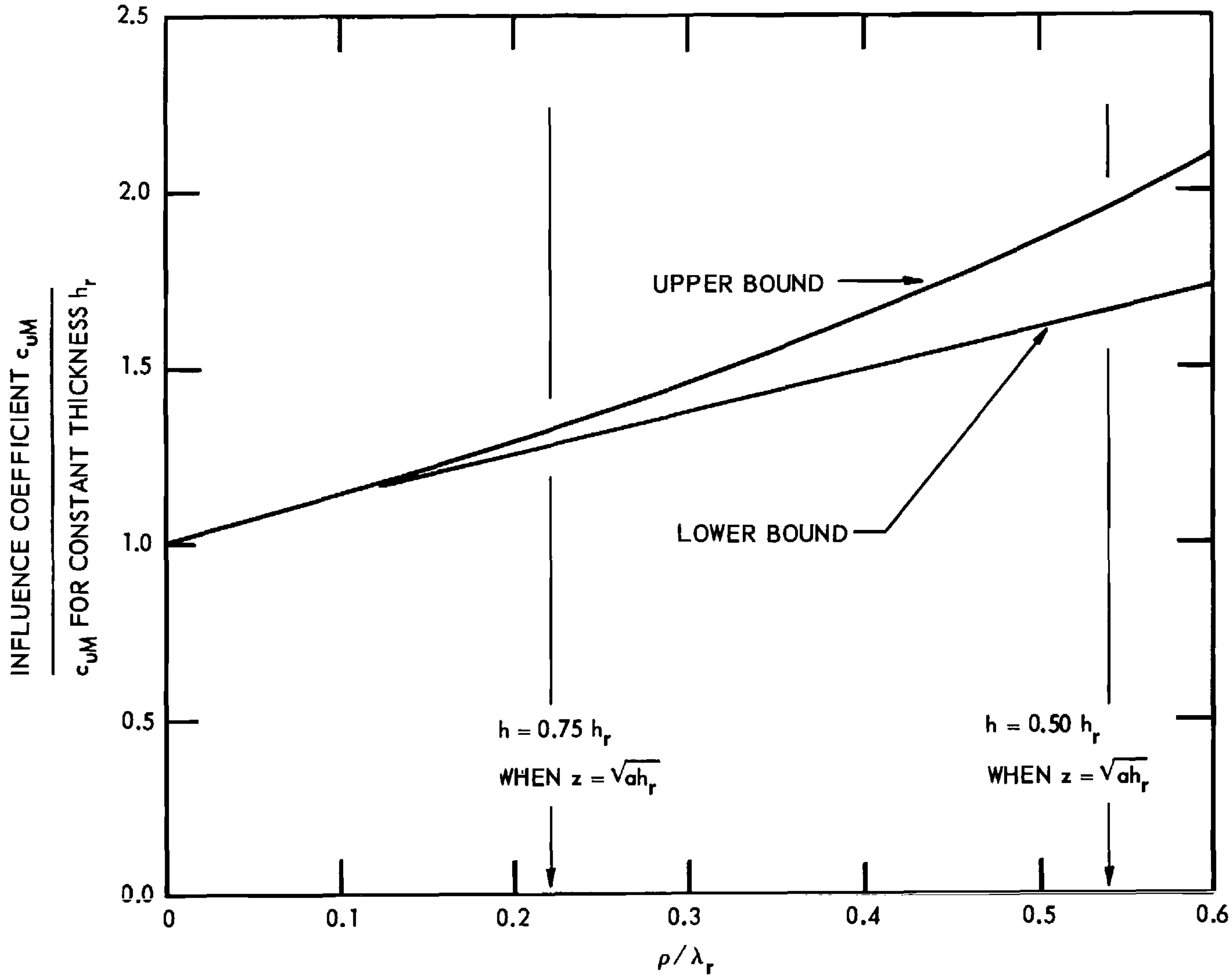
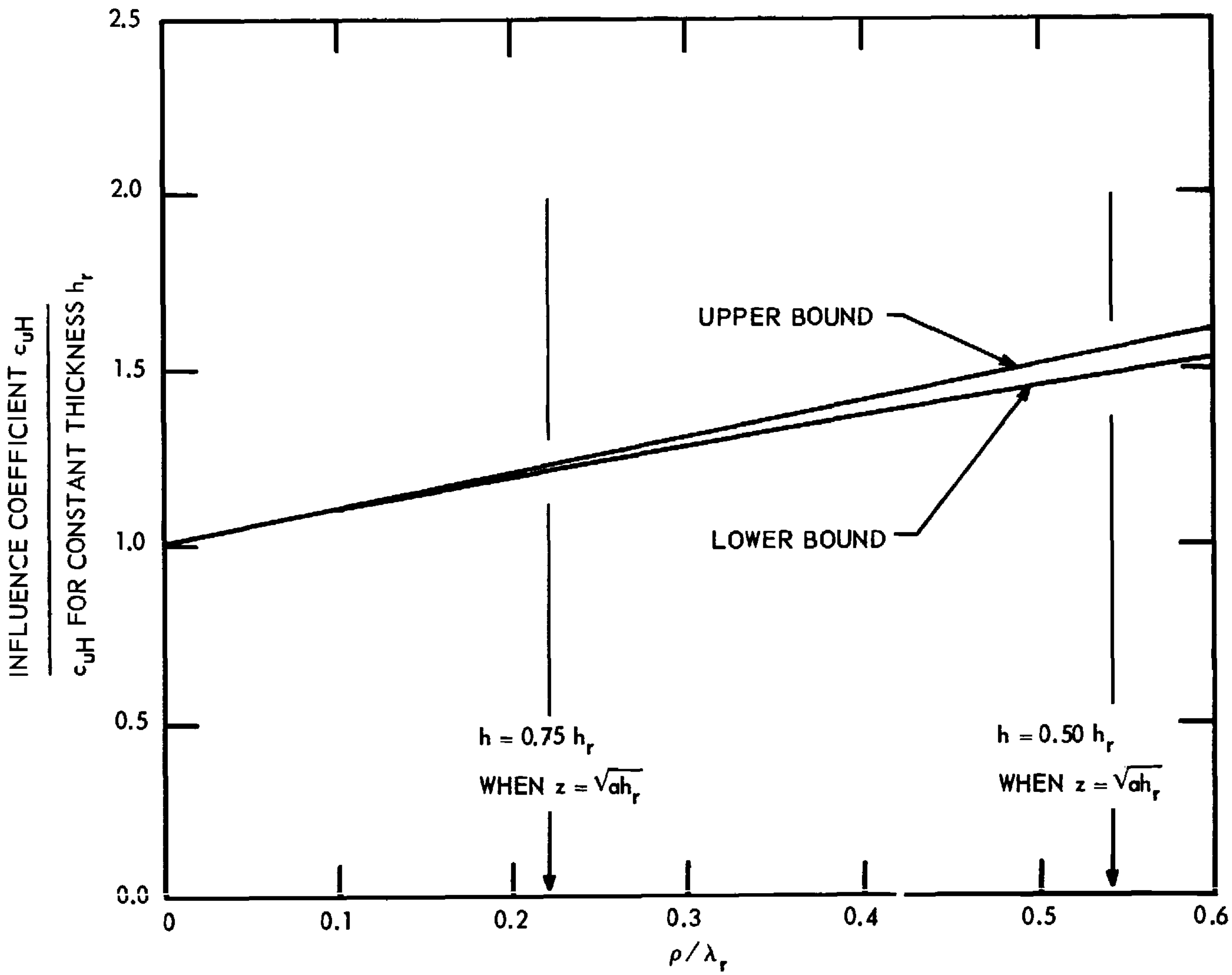

2.5
2.0
1.5
1.0
0.5
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
ρ/λ_r
INFLUENCE COEFFICIENT c_{uM} / c_{uM} FOR CONSTANT THICKNESS h_r
UPPER BOUND
LOWER BOUND
$h = 0.75\ h_r$
WHEN $z = \sqrt{ah_r}$
$h = 0.50\ h_r$
WHEN $z = \sqrt{ah_r}$

Figure 5



2.5
2.0
1.5
1.0
0.5
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
INFLUENCE COEFFICIENT c_{uH} / c_{uH} FOR CONSTANT THICKNESS h_r
UPPER BOUND
LOWER BOUND
$h = 0.75\ h_r$
WHEN $z = \sqrt{ah_r}$
$h = 0.50\ h_r$
WHEN $z = \sqrt{ah_r}$
ρ / λ_r

Figure 6

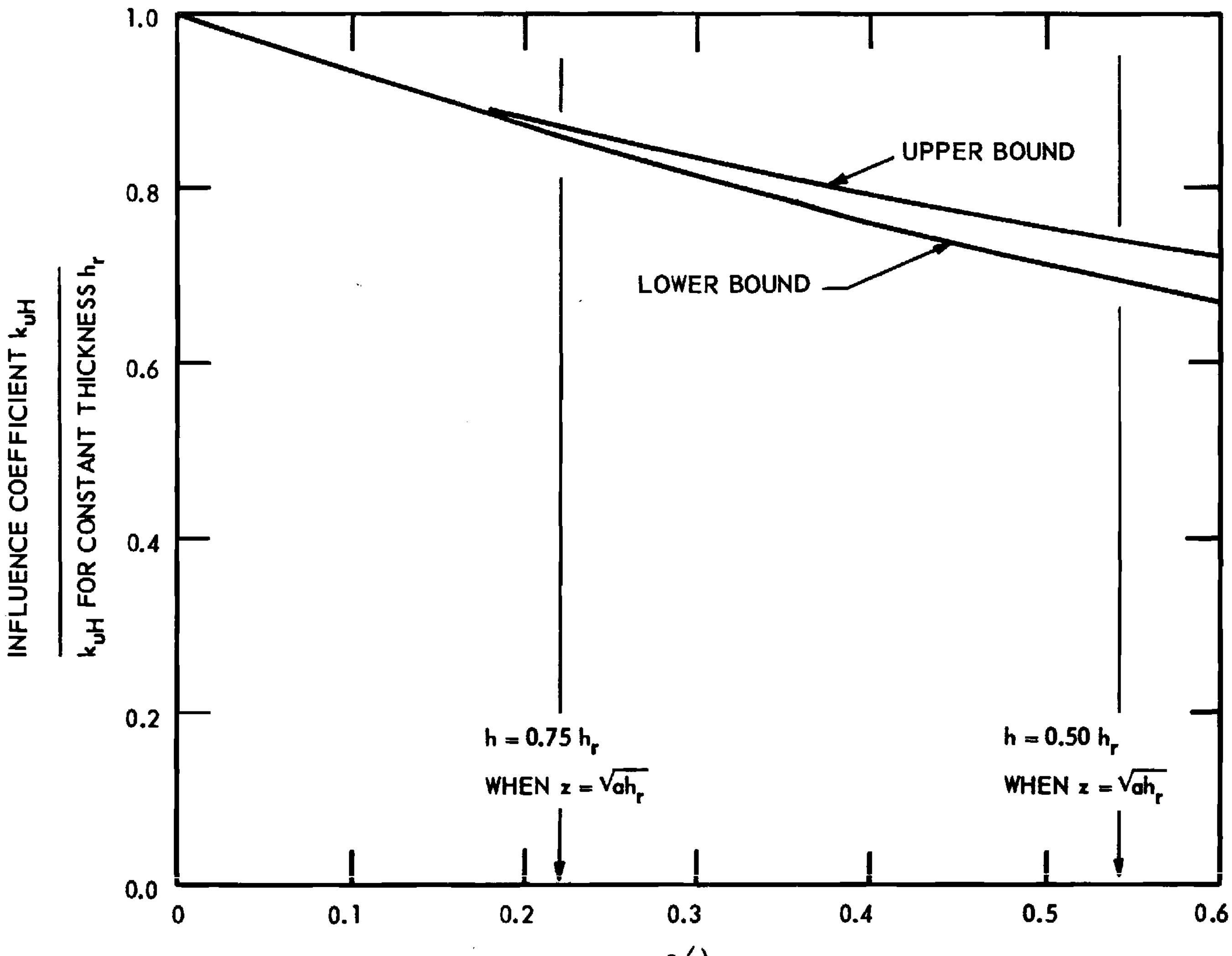
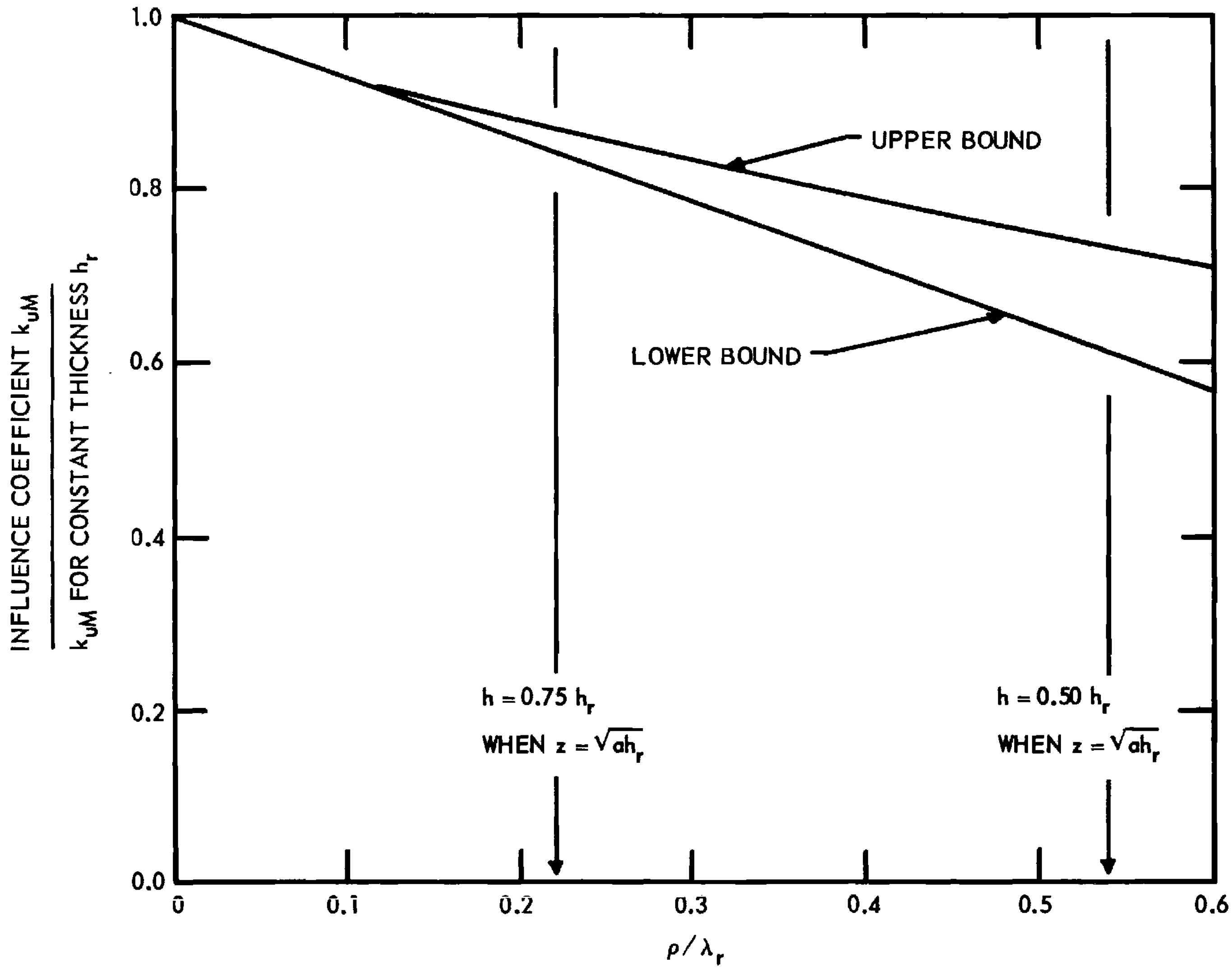
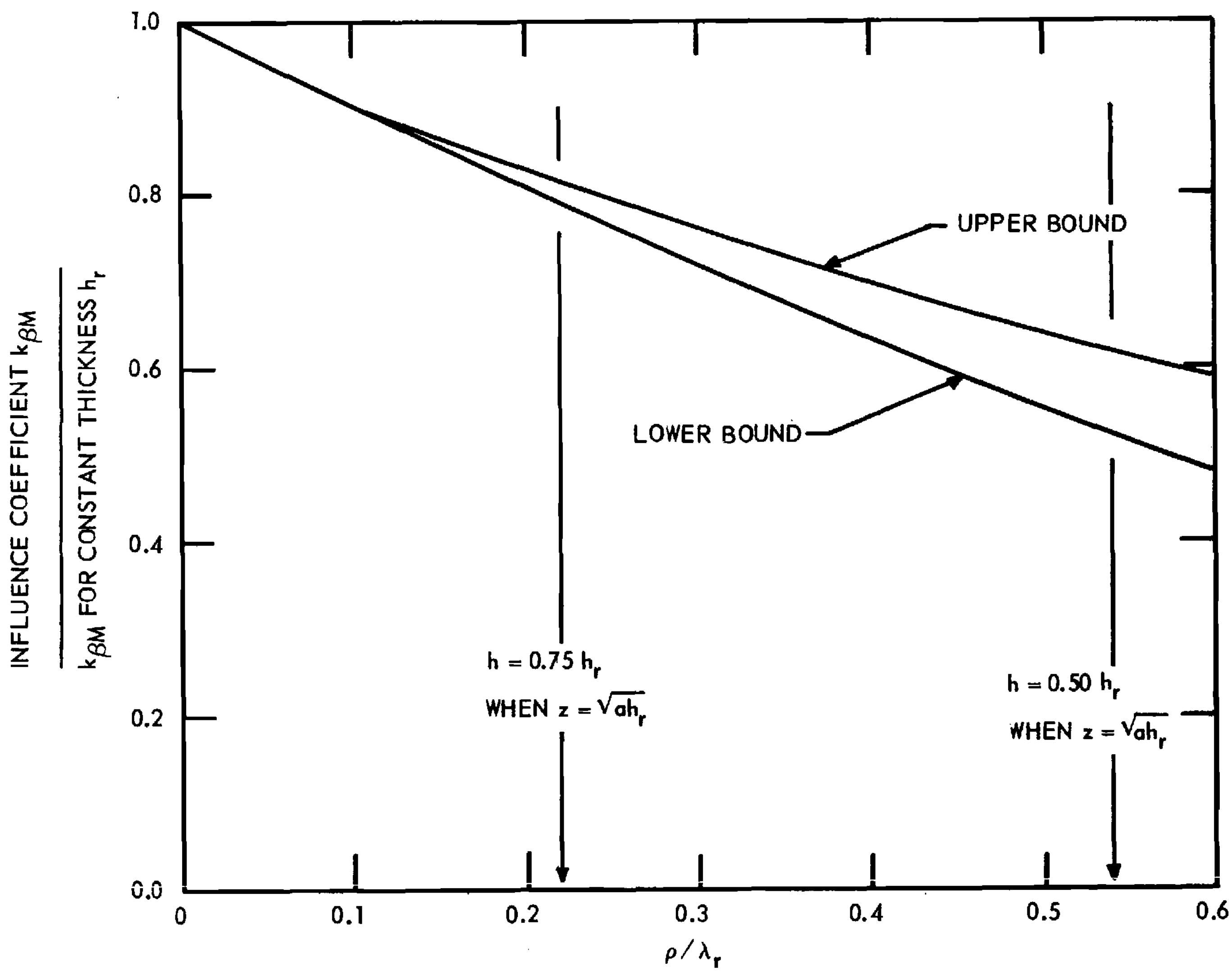

UPPER BOUND
LOWER BOUND
h = 0.75 h_r
WHEN z = $\sqrt{ah_r}$
h = 0.50 h_r
WHEN z = $\sqrt{ah_r}$
INFLUENCE COEFFICIENT k_{uH}
k_{uH} FOR CONSTANT THICKNESS h_r
1.0
0.8
0.6
0.4
0.2
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
ρ/λ_r

Figure 7



1.0
0.8
0.6
0.4
0.2
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
INFLUENCE COEFFICIENT k_{uM}
k_{uM} FOR CONSTANT THICKNESS h_r
UPPER BOUND
LOWER BOUND
$h = 0.75\ h_r$
WHEN $z = \sqrt{ah_r}$
$h = 0.50\ h_r$
WHEN $z = \sqrt{ah_r}$
ρ/λ_r

Figure 8



INFLUENCE COEFFICIENT $k_{\beta M}$
$k_{\beta M}$ FOR CONSTANT THICKNESS h_r
1.0
0.8
0.6
0.4
0.2
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
ρ / λ_r
UPPER BOUND
LOWER BOUND
$h = 0.75\ h_r$
WHEN $z = \sqrt{ah_r}$
$h = 0.50\ h_r$
WHEN $z = \sqrt{ah_r}$

Figure 9

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by

M. B. Sledd

April 10, 1957

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Influence Coefficients for Circular Cylindrical Shells
with Linearly Varying Wall Thickness*

by

M. B. Sledd

Introduction. One edge of a thin-walled elastic right circular cylindrical shell is loaded by a uniformly distributed shearing force H_2 and bending moment M_2; the other edge is free (Figure 1). The applied loads produce rotationally symmetric deformations in the shell; and if the deformations are small enough, they are linearly related (approximately) to the force H_2 and moment M_2 which produce them. In particular, the loaded edge experiences a radial displacement u_2 and a rotation β_2 which may be expressed in the form

$$u_2 = C_{uH}H_2 + C_{uM}M_2 , \tag{1}$$

$$\beta_2 = C_{\beta H}H_2 + C_{\beta M}M_2 . \tag{2}$$

The parameters C_{uH}, C_{uM} $(= C_{\beta H})$, and $C_{\beta M}$ are the influence coefficients for the shell. In the defining equations (1) and (2) the choice of signs is such that all the C's are positive. Note in this connection that the coordinate parameter $\bar{z}$ increases as the loaded edge is approached through points lying on the middle surface of the shell.

For purposes of subsequent comparison, the first section of the following discussion records expressions for the influence coefficients of the cylindrical shell of constant wall thickness and semi-infinite axial length and similar expressions for the cylindrical shell of constant wall thickness and finite axial length. Then the cylindrical shell with linearly varying wall thickness is considered; and expressions are deduced for the radial displacement u, rotation β, "horizontal" stress resultant H, and bending moment M as functions of axial distance. These expressions are used to obtain the influence coefficients for a shell loaded along its thicker edge. The results are compared in graphical form with those previously recorded for shells of constant thickness.

The cylindrical shell with linearly varying wall thickness has been discussed by several authors (1,2,3,4)**, who expressed their results in terms of power series, asymptotic series, and the so-called Schleicher functions.

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**Numbered references appear in the bibliography.

The discussion in the present paper is given in terms of the Kelvin functions ($ber_n x$, $bei_n x$, $ker_n x$, $kei_n x$), which have been extensively tabulated in recent years (5,6,7,8,9,10). In comparison with the previous discussions use of the Kelvin functions is believed to offer certain advantages in simplicity and in standardization of nomenclature.

The results for the influence coefficients of the shell with linearly varying wall thickness are new.

1. Influence coefficients for cylindrical shells of constant wall thickness. In a previous Technical Note (11) the following expressions are given for the influence coefficients of the cylindrical shell of constant wall thickness and semi-infinite axial length.

$$C_{uHr} = \frac{a^3}{2D_r\lambda_r^{\,3}} \tag{3}$$

$$C_{uMr} = C_{\beta Hr} = \frac{a^2}{2D_r\lambda_r^{\,2}} \tag{4}$$

$$C_{\beta Mr} = \frac{a}{D_r\lambda_r} \tag{5}$$

Here a is the radius of the undeformed middle surface;

$$D_r = \frac{Eh_r^{\,3}}{12(1-\nu^2)}\ , \tag{6}$$

where E is Young's modulus,
ν is Poisson's ratio,
h_r is the thickness of the shell wall;

and

$$\lambda_r = \sqrt[4]{3(1-\nu^2)}\ \sqrt{\frac{a}{h_r}}\ . \tag{7}$$

The subscript r refers to the fact that a reference thickness h_r is involved. For a shell of constant thickness, the reference thickness h_r and the thickness h at any point are evidently the same.

Esslinger (12) gives the following expressions for the influence coefficients of the cylindrical shell of constant wall thickness h_r and finite axial length L.

$$C_{uH} = (C_{uHr})\left[\frac{e^{2\eta}-e^{-2\eta}-4\sin\eta\cos\eta}{e^{2\eta}+e^{-2\eta}-4\sin^2\eta-2}\right] \tag{8}$$

$$C_{uM} = C_{\beta H} = (C_{uMr})\left[\frac{e^{2\eta}+e^{-2\eta}+4\sin^2\eta-2}{e^{2\eta}+e^{-2\eta}-4\sin^2\eta-2}\right] \tag{9}$$

$$C_{\beta M} = (C_{\beta Mr}) \left[\frac{e^{2\eta}-e^{-2\eta}+4 \sin\eta \cos\eta}{e^{2\eta}+e^{-2\eta}-4 \sin^2\eta - 2}\right] \quad (10)$$

Here $\eta = (\lambda_r L)/a$. It will be noted that the quantities in the square brackets are the ratios of the influence coefficients for the shell of finite length to those for the shell of semi-infinite length.

2. The cylindrical shell with linearly varying wall thickness. Let the geometry of the shell be described as shown in Figure 1, where z is measured positively upward from the point where the wall thickness would be zero if the shell were sufficiently extended. If the thickness of the shell at the upper edge is taken as the reference thickness h_r, then

$$h = h_r \left(\frac{z}{L_2}\right) . \quad (11)$$

The state of stress in the shell is represented by stress resultants N_z, N_θ, H and by stress couples M_z, M_θ (Figure 2). In the absence of surface loads and axially directed edge loads,

$$N_z = 0 ; \quad (12)$$

and since within the limitations of a linear theory the change in circumferential curvature is zero,

$$M_\theta = \nu M_z . \quad (13)$$

The remaining stress resultants and couple satisfy the equilibrium equations

$$\frac{dH}{dz} - \frac{N_\theta}{a} = 0 , \quad (14)$$

$$\frac{dM_z}{dz} + H = 0 . \quad (15)$$

Let u represent the radial displacement of the middle surface and β the change in inclination of a middle-surface meridian due to deformation of the shell (Figure 3). Then u and β satisfy the compatibility relation

$$\beta = \frac{du}{dz} . \quad (16)$$

The deflections u and β are accompanied by a circumferential strain ε_θ and a change K_z of meridional curvature given by the strain-displacement relations

$$\varepsilon_\theta = \frac{u}{a} , \quad (17)$$

$$K_z = \frac{d\beta}{dz} . \quad (18)$$

For an isotropic material satisfying Hooke's Law, and under the assumption that transverse shear deformation is negligible, the strains ε_θ and K_z satisfy two stress-strain relations:

$$\varepsilon_\theta = \frac{N_\theta}{C} , \tag{19}$$

$$K_z = \frac{M_z}{D} , \tag{20}$$

where $C = Eh$, $D = Eh^3/12(1-\nu^2)$, and h is the wall thickness as given by equation (11).

Equations (14) - (20) are a set of seven equations for the seven unknowns H, N_θ, M_z, u, β, ε_θ, K_z. Eliminating N_θ, M_z, u, ε_θ, and K_z leads to two second-order differential equations for β and H .

$$D \frac{d^2\beta}{dz^2} + \frac{dD}{dz}\frac{d\beta}{dz} + H = 0 \tag{21}$$

$$\frac{d^2H}{dz^2} - \frac{1}{C}\frac{dC}{dz}\frac{dH}{dz} - \frac{C}{a^2}\beta = 0 \tag{22}$$

The first-derivative terms $d\beta/dz$ and dH/dz can be eliminated by the introduction of new dependent variables X and Y , where

$$X = D^{1/2}\,\beta \tag{23}$$

and

$$Y = aC^{-1/2}\,H . \tag{24}$$

If the terms involving D, C, and their derivatives are then expressed in terms of z by use of equation (11), equations (21) and (22) become

$$\frac{d^2X}{dz^2} - \frac{3}{4}\frac{1}{z^2}X + 2\lambda_r^2\left(\frac{L_2}{a^2 z}\right) Y = 0 ; \tag{25}$$

$$\frac{d^2Y}{dz^2} - \frac{3}{4}\frac{1}{z^2}Y - 2\lambda_r^2\left(\frac{L_2}{a^2 z}\right) X = 0 . \tag{26}$$

Multiplying (26) by $i(=\sqrt{-1})$ and adding the result to (25) yields a single second-order differential equation

$$\frac{d^2Z}{dz^2} - \left[\frac{3}{4}\frac{1}{z^2} + 2i\,\lambda_r^2\left(\frac{L_2}{a^2}\right)\frac{1}{z}\right] Z = 0 \tag{27}$$

for the combined dependent variable

$$Z = X + i\,Y \,. \tag{28}$$

The general solution of equation (27) is

$$Z(x) = x\begin{Bmatrix} C_1 \operatorname{ber}_2 x - C_2 \operatorname{bei}_2 x + C_3 \operatorname{ker}_2 x - C_4 \operatorname{kei}_2 x \\ +i(C_1 \operatorname{bei}_2 x + C_2 \operatorname{ber}_2 x + C_3 \operatorname{kei}_2 x + C_4 \operatorname{ker}_2 x) \end{Bmatrix}, \tag{29}$$

where

$$x = \frac{2}{a}\,\lambda_r\sqrt{2L_2}\;z^{1/2} \,. \tag{30}$$

Separating $Z(x)$ into its real and imaginary parts gives

$$X = x(C_1 \operatorname{ber}_2 x - C_2 \operatorname{bei}_2 x + C_3 \operatorname{ker}_2 x - C_4 \operatorname{kei}_2 x) \,, \tag{31}$$

$$Y = x(C_1 \operatorname{bei}_2 x + C_2 \operatorname{ber}_2 x + C_3 \operatorname{kei}_2 x + C_4 \operatorname{ker}_2 x) \,. \tag{32}$$

From equations (23), (24), (20), (18), (14), (19), and (17), it follows by the use of known relations among the Kelvin functions and their derivatives (see Appendix I) that

$$\begin{aligned} \beta &= D^{-1/2}X \\ &= \frac{16\sqrt{2}\,\lambda_r^3}{\sqrt{D_r}}\left(\frac{L_2}{a}\right)^3 x^{-2}(C_1 \operatorname{ber}_2 x - C_2 \operatorname{bei}_2 x + C_3 \operatorname{ker}_2 x - C_4 \operatorname{kei}_2 x) \,; \end{aligned} \tag{33}$$

$$\begin{aligned} H &= a^{-1}\,C^{1/2}\,Y \\ &= \frac{\lambda_r\sqrt{D_r}}{a^2\sqrt{2}}\left(\frac{a}{L_2}\right) x^2\,(C_1 \operatorname{bei}_2 x + C_2 \operatorname{ber}_2 x + C_3 \operatorname{kei}_2 x + C_4 \operatorname{ker}_2 x) \,; \end{aligned} \tag{34}$$

$$\begin{aligned} M_z &= D\,\frac{d\beta}{dz} = \left(D\,\frac{dx}{dz}\right)\frac{d\beta}{dx} \\ &= \frac{\sqrt{D_r}}{4\sqrt{2}\,a\lambda_r}\left(\frac{a}{L_2}\right)^2 x^3 \begin{Bmatrix} C_1\left(\operatorname{ber}_2' x - \frac{2}{x}\operatorname{ber}_2 x\right) - C_2\left(\operatorname{bei}_2' x - \frac{2}{x}\operatorname{bei}_2 x\right) \\ +C_3\left(\operatorname{ker}_2' x - \frac{2}{x}\operatorname{ker}_2 x\right) - C_4\left(\operatorname{kei}_2' x - \frac{2}{x}\operatorname{kei}_2 x\right) \end{Bmatrix} \end{aligned}$$

$$= \frac{\sqrt{D_r}}{8a\lambda_r} \left(\frac{a}{L_2}\right)^2 x^3 \left\{ \begin{array}{l} C_1(\mathrm{ber}_3\, x + \mathrm{bei}_3\, x) + C_2(\mathrm{ber}_3\, x - \mathrm{bei}_3\, x) \\ +C_3(\mathrm{ker}_3\, x + \mathrm{kei}_3\, x) + C_4(\mathrm{ker}_3\, x - \mathrm{kei}_3\, x) \end{array} \right\} ; \tag{35}$$

$$u = \frac{a^2}{C} \frac{dH}{dz} = \left(\frac{a^2}{C} \frac{dx}{dz}\right) \frac{dH}{dx}$$

$$= \frac{4\sqrt{2}a\lambda_r}{\sqrt{D_r}} \left(\frac{L_2}{a}\right)^2 \left(\frac{1}{x}\right) \left\{ \begin{array}{l} C_1(\mathrm{bei}_2'\, x + \frac{2}{x}\,\mathrm{bei}_2\, x) + C_2(\mathrm{ber}_2'\, x + \frac{2}{x}\,\mathrm{ber}_2\, x) \\ +C_3(\mathrm{kei}_2'\, x + \frac{2}{x}\,\mathrm{kei}_2\, x) + C_4(\mathrm{ker}_2'\, x + \frac{2}{x}\,\mathrm{ker}_2\, x) \end{array} \right\}$$

$$= \frac{4a\lambda_r}{\sqrt{D_r}} \left(\frac{L_2}{a}\right)^2 \left(\frac{1}{x}\right) \left\{ \begin{array}{l} C_1(\mathrm{ber}_1\, x - \mathrm{bei}_1\, x) - C_2(\mathrm{ber}_1\, x + \mathrm{bei}_1\, x) \\ +C_3(\mathrm{ker}_1\, x - \mathrm{kei}_1\, x) - C_4(\mathrm{ker}_1\, x + \mathrm{kei}_1\, x) \end{array} \right\} . \tag{36}$$

3. <u>Influence coefficients for the shell with vanishing wall thickness at one edge.</u> As a limiting case not without some practical interest, suppose that the shell in Figure 1 is extended downward to a knife edge so that the dimension L_1 is zero (then L, the actual length of the shell, is the same as L_2). Suppose that the lower edge of the shell is free and that to the upper edge are applied a bending moment M_2 and a transverse shear H_2 per unit circumferential length.

If equations (33) and (36) are to yield finite displacements u and β along the lower edge, it is necessary that

$$C_3 = C_4 = 0 ,$$

which in turn insures that the lower edge is free [from equations (34) and (35), $H(0) = M_z(0) = 0$].

Setting $H = H_2$ and $M_z = M_2$ at $z = L_2$ (i.e., at $x = x_2 = \frac{2\sqrt{2}}{a} \lambda_r L_2$) in equations (34) and (35) yields

$$C_1\, \mathrm{bei}_2\, x_2 + C_2\, \mathrm{ber}_2\, x_2 = \frac{a^2}{4\sqrt{2}\,\lambda_r^3 \sqrt{D_r}} \left(\frac{a}{L_2}\right) H_2 ; \tag{37}$$

$$C_1(\mathrm{ber}_3\, x_2 + \mathrm{bei}_3\, x_2) + C_2(\mathrm{ber}_3\, x_2 - \mathrm{bei}_3\, x_2) = \frac{a}{2\sqrt{2}\,\lambda_r^2 \sqrt{D_r}} \left(\frac{a}{L_2}\right) M_2 . \tag{38}$$

Solving these equations for C_1 and C_2, introducing the values so found in equations (33) and (36), and then setting $x = x_2 = (2\sqrt{2}\lambda_r L_2)/a$ in the results gives the following expressions for the displacements u and β at the loaded edge:

$$u(x_2) = \left(\frac{a^3}{2D_r\lambda_r^3}\right)\left[\frac{\mathrm{ber}_3x_2\mathrm{ber}_1x_2 + \mathrm{bei}_3x_2\mathrm{bei}_1x_2}{\Delta_2}\right] H_2$$

$$+ \left(\frac{a^2}{2D_r\lambda_r^2}\right)\left[\frac{-\mathrm{ber}_2x_2(\mathrm{ber}_1x_2 - \mathrm{bei}_1x_2) - \mathrm{bei}_2x_2(\mathrm{ber}_1x_2 + \mathrm{bei}_1x_2)}{\Delta_2}\right] M_2 , \tag{39}$$

$$\beta(x_2) = \left(\frac{a^2}{2D_r\lambda_r^2}\right)\left[\frac{-\mathrm{ber}_2x_2(\mathrm{ber}_1x_2 - \mathrm{bei}_1x_2) - \mathrm{bei}_2x_2(\mathrm{ber}_1x_2 + \mathrm{bei}_1x_2)}{\Delta_2}\right] H_2$$

$$+ \left(\frac{a}{D_r\lambda_r}\right)\left[-\frac{\mathrm{ber}_2^2\, x_2 + \mathrm{bei}_2^2\, x_2}{\Delta_2}\right] M_2 , \tag{40}$$

where

$$\Delta_2 = \frac{4\sqrt{2}}{x_2}(\mathrm{ber}_2^2\, x_2 + \mathrm{bei}_2^2\, x_2) + \frac{x_2}{\sqrt{2}}(\mathrm{bei}_2\, x_2\, \mathrm{ber}\, x_2 - \mathrm{ber}_2\, x_2\, \mathrm{bei}\, x_2) . \tag{41}$$

Comparison of equations (39) and (40) with equations (1) and (2) shows that

$$C_{uH} = \left(\frac{a^3}{2D_r\lambda_r^3}\right)\left[\frac{\mathrm{ber}_3x_2\ \mathrm{ber}_1x_2 + \mathrm{bei}_3x_2\ \mathrm{bei}_1x_2}{\Delta_2}\right] ; \tag{42}$$

$$C_{uM} = C_{\beta H} = \left(\frac{a^2}{2D_r\lambda_r^2}\right)\left[\frac{-\mathrm{ber}_2x_2\ (\mathrm{ber}_1x_2 - \mathrm{bei}_1x_2) - \mathrm{bei}_2x_2\ (\mathrm{ber}_1x_2 + \mathrm{bei}_1x_2)}{\Delta_2}\right] ; \tag{43}$$

$$C_{\beta M} = \left(\frac{a}{D_r\lambda_r}\right)\left[-\frac{\mathrm{ber}_2^2\, x_2 + \mathrm{bei}_2^2\, x_2}{\Delta_2}\right] . \tag{44}$$

In equations (42) - (44), each of the influence coefficients is written as the product of a term in parentheses by a term in square brackets. The terms in parentheses are the relevant influence coefficients for the semi-infinite cylindrical shell of constant thickness. As should then be the case, the terms in square brackets approach unity as L_2 approaches infinity.

For finite L_2 the square-bracketed terms represent the factors by which the actual influence coefficients exceed those for the semi-infinite shell of constant thickness. The excess is due (1) to the finite length of the shell and (2) to the taper of the wall. How much of the total increase is attributable to each cause separately is illustrated in Figures 4 - 6.

4. <u>Influence coefficients for the shell with non-zero wall thickness at each edge.</u> Suppose in Figure 1 that L_1 is positive. The actual length L of the shell is then $L_2 - L_1$. Suppose that the thinner edge is free and that the thicker edge is loaded by a bending moment M_2 and a transverse shearing force H_2.

The appropriate boundary conditions are

$$z = L_1 \;\left(x = x_1 = 2\sqrt{2}\,\lambda_r \frac{\sqrt{L_1 L_2}}{a}\right) \;:\quad H = 0 \;;$$

$$z = L_1 \;\left(x = x_1 = 2\sqrt{2}\,\lambda_r \frac{\sqrt{L_1 L_2}}{a}\right) \;:\quad M_z = 0 \;;$$

$$z = L_2 \;\left(x = x_2 = 2\sqrt{2}\,\lambda_r \frac{L_2}{a}\right) \;:\quad H = H_2 \;;$$

$$z = L_2 \;\left(x = x_2 = 2\sqrt{2}\,\lambda_r \frac{L_2}{a}\right) \;:\quad M_z = M_2 \;.$$

Applying these boundary conditions in equations (34) and (35) gives four equations for the determination of the constants C_1, C_2, C_3, C_4. Thus,

$$C_1 \operatorname{bei}_2 x_1 + C_2 \operatorname{ber}_2 x_1 + C_3 \operatorname{kei}_2 x_1 + C_4 \operatorname{ker}_2 x_1 = 0 \;; \tag{45}$$

$$C_1 \left(\frac{\operatorname{ber}_3 x_1 + \operatorname{bei}_3 x_1}{\sqrt{2}}\right) + C_2 \left(\frac{\operatorname{ber}_3 x_1 - \operatorname{bei}_3 x_1}{\sqrt{2}}\right) + C_3 \left(\frac{\operatorname{ker}_3 x_1 + \operatorname{kei}_3 x_1}{\sqrt{2}}\right) + C_4 \left(\frac{\operatorname{ker}_3 x_1 - \operatorname{kei}_3 x_1}{\sqrt{2}}\right) = 0 \;; \tag{46}$$

$$C_1 \operatorname{bei}_2 x_2 + C_2 \operatorname{ber}_2 x_2 + C_3 \operatorname{kei}_2 x_2 + C_4 \operatorname{ker}_2 x_2 = \frac{a^2}{4\sqrt{2}\,\lambda_r^3 \sqrt{D_r}} \left(\frac{a}{L_2}\right) H_2 \;; \tag{47}$$

$$C_1 \left(\frac{\operatorname{ber}_3 x_2 + \operatorname{bei}_3 x_2}{\sqrt{2}}\right) + C_2 \left(\frac{\operatorname{ber}_3 x_2 - \operatorname{bei}_3 x_2}{\sqrt{2}}\right) + C_3 \left(\frac{\operatorname{ker}_3 x_2 + \operatorname{kei}_3 x_2}{\sqrt{2}}\right) + C_4 \left(\frac{\operatorname{ker}_3 x_2 - \operatorname{kei}_3 x_2}{\sqrt{2}}\right) = \frac{a}{4\,\lambda_r^2 \sqrt{D_r}} \left(\frac{a}{L_2}\right) M_2 \;. \tag{48}$$

In equations (45) - (48) let Δ represent the determinant of the coefficients of the C's. Thus,

$$\Delta = \begin{vmatrix} \mathrm{bei}_2 x_1 & \mathrm{ber}_2 x_1 & \mathrm{kei}_2 x_1 & \mathrm{ker}_2 x_1 \\ \frac{1}{\sqrt{2}}(\mathrm{ber}_3 x_1 + \mathrm{bei}_3 x_1) & \frac{1}{\sqrt{2}}(\mathrm{ber}_3 x_1 - \mathrm{bei}_3 x_1) & \frac{1}{\sqrt{2}}(\mathrm{ker}_3 x_1 + \mathrm{kei}_3 x_1) & \frac{1}{\sqrt{2}}(\mathrm{ker}_3 x_1 - \mathrm{kei}_3 x_1) \\ \mathrm{bei}_2 x_2 & \mathrm{ber}_2 x_2 & \mathrm{kei}_2 x_2 & \mathrm{ker}_2 x_2 \\ \frac{1}{\sqrt{2}}(\mathrm{ber}_3 x_2 + \mathrm{bei}_3 x_2) & \frac{1}{\sqrt{2}}(\mathrm{ber}_3 x_2 - \mathrm{bei}_3 x_2) & \frac{1}{\sqrt{2}}(\mathrm{ker}_3 x_2 + \mathrm{kei}_3 x_2) & \frac{1}{\sqrt{2}}(\mathrm{ker}_3 x_2 - \mathrm{kei}_3 x_2) \end{vmatrix} . \tag{49}$$

Let A_{ij} be the third-order determinant obtained from Δ by deleting the ith row and j^{th} column. Solving equations (45) - (48) for C_1, C_2, C_3, C_4 then gives

$$C_1 = \frac{a^2}{4\sqrt{2}\,\lambda_r^3\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) H_2\,(A_{31}) - \frac{a}{4\,\lambda_r^2\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) M_2\,(A_{41})\ ;$$

$$C_2 = -\frac{a^2}{4\sqrt{2}\,\lambda_r^3\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) H_2\,(A_{32}) + \frac{a}{4\,\lambda_r^2\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) M_2\,(A_{42})\ ;$$

$$C_3 = \frac{a^2}{4\sqrt{2}\,\lambda_r^3\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) H_2\,(A_{33}) - \frac{a}{4\,\lambda_r^2\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) M_2\,(A_{43})\ ;$$

$$C_4 = -\frac{a^2}{4\sqrt{2}\,\lambda_r^3\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) H_2\,(A_{34}) + \frac{a}{4\,\lambda_r^2\,\Delta\sqrt{D_r}}\left(\frac{a}{L_2}\right) M_2\,(A_{44})\ .$$

When these expressions for the C's are substituted in equations (33) and (36) with $x = x_2$, the expressions for $u(x_2)$ and $\beta(x_2)$ are found to be

$$u(x_2) = \frac{a^3}{2D_r\lambda_r^3}\left(\frac{1}{\Delta\sqrt{2}}\right)\left\{\begin{matrix} A_{31}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 - \mathrm{bei}_1 x_2)] + A_{32}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 + \mathrm{bei}_1 x_2)] \\ + A_{33}[\frac{1}{\sqrt{2}}[\mathrm{ker}_1 x_2 - \mathrm{kei}_1 x_2)] + A_{34}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 + \mathrm{kei}_1 x_2)] \end{matrix}\right\} H_2$$

$$+ \frac{a^2}{2D_r\lambda_r^2}\left(\frac{1}{\Delta}\right)\left\{\begin{array}{l} - A_{41}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 - \mathrm{bei}_1 x_2)] - A_{42}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 + \mathrm{bei}_1 x_2)] \\ - A_{43}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 - \mathrm{kei}_1 x_2)] - A_{44}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 + \mathrm{kei}_1 x_2)] \end{array}\right\} M_2 \; ; \tag{50}$$

$$\beta(x_2) = \frac{a^2}{2D_r\lambda_r^2}\left(\frac{1}{\Delta}\right)\left\{\begin{array}{l} A_{31}\ \mathrm{ber}_2 x_2 + A_{32}\ \mathrm{bei}_2 x_2 \\ + A_{33}\ \mathrm{ker}_2 x_2 + A_{34}\ \mathrm{kei}_2 x_2 \end{array}\right\} H_2$$

$$+ \frac{a}{D_r\lambda_r}\left(\frac{1}{\Delta\sqrt{2}}\right)\left\{\begin{array}{l} -A_{41}\ \mathrm{ber}_2 x_2 - A_{42}\ \mathrm{bei}_2 x_2 \\ -A_{43}\ \mathrm{ker}_2 x_2 - A_{44}\ \mathrm{kei}_2 x_2 \end{array}\right\} M_2 \; . \tag{51}$$

Comparison of equations (50) and (51) with equations (1) and (2) gives the following expressions for the influence coefficients.

$$C_{uH} = \left(\frac{a^3}{2D_r\lambda_r^3}\right)\left(\frac{1}{\Delta\sqrt{2}}\right)\left\{\begin{array}{l} A_{31}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 - \mathrm{bei}_1 x_2)] + A_{32}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 + \mathrm{bei}_1 x_2)] \\ + A_{33}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 - \mathrm{kei}_1 x_2)] + A_{34}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 + \mathrm{kei}_1 x_2)] \end{array}\right\} \tag{52}$$

$$C_{uM} = \left(\frac{a^2}{2D_r\lambda_r^2}\right)\left(-\frac{1}{\Delta}\right)\left\{\begin{array}{l} A_{41}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 - \mathrm{bei}_1 x_2)] + A_{42}[\frac{1}{\sqrt{2}}(\mathrm{ber}_1 x_2 + \mathrm{bei}_1 x_2)] \\ + A_{43}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 - \mathrm{kei}_1 x_2)] + A_{44}[\frac{1}{\sqrt{2}}(\mathrm{ker}_1 x_2 + \mathrm{kei}_1 x_2)] \end{array}\right\} \tag{53}$$

$$C_{\beta H} = \left(\frac{a^2}{2D_r\lambda_r^2}\right)\left(\frac{1}{\Delta}\right)(A_{31}\ \mathrm{ber}_2 x_2 + A_{32}\ \mathrm{bei}_2 x_2 + A_{33}\ \mathrm{ker}_2 x_2 + A_{34}\ \mathrm{kei}_2 x_2) \tag{54}$$

$$C_{\beta M} = \left(\frac{a}{D_r\lambda_r}\right)\left(-\frac{1}{\Delta\sqrt{2}}\right)(A_{41}\ \mathrm{ber}_2 x_2 + A_{42}\ \mathrm{bei}_2 x_2 + A_{43}\ \mathrm{ker}_2 x_2 + A_{44}\ \mathrm{kei}_2 x_2) \tag{55}$$

Here $x_2 = (2\sqrt{2}\,\lambda_r L_2)/a$; Δ is defined by equation (49); and A_{ij} is the minor (not the cofactor) of the element in the i^{th} row and j^{th} column of Δ.

It may be verified by direct algebraic calculation that the expressions for C_{uM} [eq. (53)] and $C_{\beta H}$ [eq. (54)] are equal.

5. Discussion of results. The curves of Figures 4 - 6 present information on the influence coefficients of three different shells in comparison with the influence coefficients of a hypothetical semi-infinite shell of constant thickness. In each figure the straight horizontal line numbered (1) represents the result for the hypothetical semi-infinite shell, which serves as a reference.

Among the three shells of finite length two serve as limiting cases; the third is intermediate between the two extremes. The curves numbered (2) apply to a shell of finite length L and constant wall thickness h_r (no taper). The curves numbered (4) apply to a shell of the same length L but with wall thickness decreasing linearly from h_r to zero (maximum taper). Intermediate between (2) and (4), the curves numbered (3) refer to a shell in which the wall thickness decreases from h_r to 0.5 h_r over the same axial distance L (half the maximum taper for a shell of the specified length loaded along the thicker edge).

In each of the cases (3) and (4) the results apply to shells loaded along the thicker edge. The method of calculation outlined in section 4, above, can be applied equally well to shells loaded along the thinner edge provided the boundary conditions are suitably modified. This modification has not been considered in the present report.

Examination of the curves numbered (2) in Figures 4 - 6 shows that so far as the influence coefficients are concerned, the length L of the shell of constant thickness is effectively infinite as soon as

$$\frac{\lambda_r L}{a} \geq 2.5 \quad .$$

Since

$$\frac{\lambda_r L}{a} = \sqrt[4]{3(1-\nu^2)}\ \frac{L}{\sqrt{a h_r}} \approx 1.25\ \frac{L}{\sqrt{a h_r}} \quad ,$$

the condition $(\lambda_r L/a) \geq 2.5$ can also be written

$$1.25\ \frac{L}{\sqrt{a h_r}} \geq 2.5$$

or

$$L \geq 2\sqrt{a h_r} \quad .$$

This result is in accord with the fact that the important bending effects occur in a narrow edge zone.

Comparison of the curves numbered (2) with the straight lines (1) gives an indication of the effect of finite length in increasing the influence coefficients above the values which they have for the semi-infinite shell. Comparison of curves (3) and (4) with (2) gives an indication of the effect of the linearly decreasing wall thickness.

Appendix I

The Kelvin Functions $ber_n x$, $bei_n x$, $ker_n x$, $kei_n x$
($n = 1, 2, 3$; x real and non-negative)

1. Derivative relations. Between the derivative of a Kelvin function of the second order and the functions of next adjacent orders, the following relations hold.

$$\sqrt{2}\,(ber_2' x + \frac{2}{x} ber_2 x) = - ber_1 x - bei_1 x$$

$$\sqrt{2}\,(ber_2' x - \frac{2}{x} ber_2 x) = ber_3 x + bei_3 x$$

$$\sqrt{2}\,(bei_2' x + \frac{2}{x} bei_2 x) = ber_1 x - bei_1 x$$

$$\sqrt{2}\,(bei_2' x - \frac{2}{x} bei_2 x) = - ber_3 x + bei_3 x$$

$$\sqrt{2}\,(ker_2' x + \frac{2}{x} ker_2 x) = - ker_1 x - kei_1 x$$

$$\sqrt{2}\,(ker_2' x - \frac{2}{x} ker_2 x) = ker_3 x + kei_3 x$$

$$\sqrt{2}\,(kei_2' x + \frac{2}{x} kei_2 x) = ker_1 x - kei_1 x$$

$$\sqrt{2}\,(kei_2' x - \frac{2}{x} kei_2 x) = - ker_3 x + kei_3 x$$

2. Behavior for small x. Retention of leading terms of the appropriate series yields approximate expressions describing the behavior of the several functions for small values of x. In the list below the symbol $\doteq$ is used to mean that the ratio of the expressions connected by it approaches unity as x approaches zero. In several instances more than one term is included, because in adding and subtracting the functions combinations at times appear in which the first addend has been canceled out. For example, as x approaches zero, $ber_1 x$ and $bei_1 x$ are $O(x)$; but the combination $ber_1 x + bei_1 x$ is $O(x^3)$.

$$ber_1 x \doteq \frac{1}{2\sqrt{2}}\left(- x - \frac{x^3}{8}\right) \qquad ker_1 x \doteq \frac{1}{\sqrt{2}}\left(- \frac{1}{x} + \frac{x \log x}{2}\right)$$

$$bei_1 x \doteq \frac{1}{2\sqrt{2}}\left(x - \frac{x^3}{8}\right) \qquad kei_1 x \doteq \frac{1}{\sqrt{2}}\left(- \frac{1}{x} - \frac{x \log x}{2}\right)$$

$$ber_2 x \doteq \frac{x^4}{96} \qquad ker_2 x \doteq \frac{1}{2}$$

$$\mathrm{bei}_2 x \doteq -\frac{x^2}{8} \qquad \mathrm{kei}_2 x \doteq \frac{2}{x^2}$$

$$\mathrm{ber}_3 x \doteq \frac{x^3}{48\sqrt{2}} \qquad \mathrm{ker}_3 x \doteq \frac{4\sqrt{2}}{x^3} - \frac{1}{x\sqrt{2}}$$

$$\mathrm{bei}_3 x \doteq \frac{x^3}{48\sqrt{2}} \qquad \mathrm{kei}_3 x \doteq -\frac{4\sqrt{2}}{x^3} - \frac{1}{x\sqrt{2}}$$

Each of the six "ber" and "bei" functions vanishes at $x = 0$. As x approaches zero, the corresponding "ker" and "kei" functions become unboundedly large, except for $\mathrm{ker}_2 x$, which approaches $1/2$.

3. Behavior for large x. As x approaches infinity, all twelve of these functions oscillate - the "ber" and "bei" functions with exponentially increasing amplitude, the "ker" and "kei" functions with exponentially decreasing amplitude.

4. Tabulated values. Equations (33) - (36) of the preceding discussion involve the functions $\mathrm{ber}_2 x$, $\mathrm{bei}_2 x$, $\mathrm{ker}_2 x$, $\mathrm{kei}_2 x$ together with their first derivatives, each of the latter being expressible in terms of the function differentiated and corresponding functions of the first or third order [this Appendix, paragraph 1].

Convenient tables for the evaluation of these functions have been prepared by Tölke (5). Values of $\mathrm{ber}_2 x$, $\mathrm{bei}_2 x$, $\mathrm{ker}_2 x$, $\mathrm{kei}_2 x$ may be read directly to four significant figures at intervals of 0.01 from 0 to 21 inclusive. The functions of first and third orders are determinable as $1/\sqrt{2}$ times the sum or difference of the entries in two parallel columns. When the Tölke tables are used, however, explicit introduction of the functions of first or third order to determine the derivative of a function of the second order is unnecessary, since provision is made in the tables for calculating $\mathrm{ber}'_2 x$, $\mathrm{bei}'_2 x$, $\mathrm{ker}'_2 x$, $\mathrm{kei}'_2 x$ as half the difference of two parallel columns. The relations between Tölke's quantities and the Kelvin functions are as follows.

Tölke	Kelvin
$-J_{11}$, J_{12}, $-G_{11}$, G_{12}	$\mathrm{ber}_1 x$, $\mathrm{bei}_1 x$, $\mathrm{ker}_1 x$, $\mathrm{kei}_1 x$
J_{21}, $-J_{22}$, G_{21}, $-G_{22}$	$\mathrm{ber}_2 x$, $\mathrm{bei}_2 x$, $\mathrm{ker}_2 x$, $\mathrm{kei}_2 x$
$-J_{31}$, J_{32}, $-G_{31}$, G_{32}	$\mathrm{ber}_3 x$, $\mathrm{bei}_3 x$, $\mathrm{ker}_3 x$, $\mathrm{kei}_3 x$

Shorter tables have been published by Dwight (6) and by McLachlan (7) in which the values of $\mathrm{ber}_n x$, $\mathrm{bei}_n x$, $\mathrm{ker}_n x$, $\mathrm{kei}_n x$ and their first derivatives are given for positive integral values of x from 1 to 10 inclusive. In these tables, $n = 1, 2, 3, 4, 5$.

Since the tables by Tölke may not be easily available and since those of Dwight and of McLachlan are hardly adequate for detailed numerical work, the following alternative is suggested.

Jahnke and Emde (8) give ber x, bei x, ker x, and kei x (3 to 5 significant figures) for values of x from 0 to 6 at intervals of 0.01 and from 6 to 10 at intervals of 0.1. $\mathrm{Ber}_1 x$, $\mathrm{bei}_1 x$, $\mathrm{ker}_1 x$, and $\mathrm{kei}_1 x$ can be obtained from their data by a simple calculation. In their notation

$$+\ \mathrm{ber}\ x = \mathrm{Re}\ J_o(x\sqrt{i})\ ;$$

$$-\ \mathrm{bei}\ x = \mathrm{Im}\ J_o(x\sqrt{i})\ ;$$

$$-\frac{2}{\pi}\ \mathrm{ker}\ x = \mathrm{Im}\ H_o^{(1)}(x\sqrt{i})\ ;$$

$$-\frac{2}{\pi}\ \mathrm{kei}\ x = \mathrm{Re}\ H_o^{(1)}(x\sqrt{i})\ ;$$

$$\frac{\sqrt{2}}{2}\ (\mathrm{ber}_1 x + \mathrm{bei}_1 x) = \frac{d}{dx}\ \mathrm{Re}\ J_o(x\sqrt{i})\ ;$$

$$\frac{\sqrt{2}}{2}\ (\mathrm{ber}_1 x - \mathrm{bei}_1 x) = \frac{d}{dx}\ \mathrm{Im}\ J_o(x\sqrt{i})\ ;$$

$$\frac{\sqrt{2}}{\pi}\ (\mathrm{ker}_1 x - \mathrm{kei}_1 x) = \frac{d}{dx}\ \mathrm{Re}\ H_o^{(1)}(x\sqrt{i})\ ;$$

$$\frac{\sqrt{2}}{\pi}\ (-\ \mathrm{ker}_1 x - \mathrm{kei}_1 x) = \frac{d}{dx}\ \mathrm{Im}\ H_o^{(1)}(x\sqrt{i})\ .$$

Similarly, references (9) and (10) give ber x, bei x, $\mathrm{ber}_1 x$, $\mathrm{bei}_1 x$ to ten decimal places at intervals of 0.01 from 0 to 10 inclusive; and ker x, kei x, $\mathrm{ker}_1 x$, $\mathrm{kei}_1 x$ are obtainable by an easy calculation. In the notation of reference (10),

$$\mathrm{ber}\ x = u_o(x, \tfrac{\pi}{4})\ ; \qquad \mathrm{bei}\ x = -\ v_o(x, \tfrac{\pi}{4})\ ;$$

$$\mathrm{ber}_1 x = -\ u_1(x, \tfrac{\pi}{4})\ ; \qquad \mathrm{bei}_1 x = v_1(x, \tfrac{\pi}{4})\ ;$$

$$\mathrm{ker}\ x = -\frac{\pi}{2}\ [U_o(x, \tfrac{\pi}{4}) + v_o(x, \tfrac{\pi}{4})]\ ;$$

$$\mathrm{kei}\ x = \frac{\pi}{2}\ [V_o(x, \tfrac{\pi}{4}) - u_o(x, \tfrac{\pi}{4})]\ ;$$

$$\mathrm{ker}_1 x = \frac{\pi}{2}\ [U_1(x, \tfrac{\pi}{4}) + v_1(x, \tfrac{\pi}{4})]\ ;$$

$$\mathrm{kei}_1 x = \frac{\pi}{2}\ [-V_1(x, \tfrac{\pi}{4}) + u_1(x, \tfrac{\pi}{4})]\ .$$

With the functions of order zero and one available from the tables, the functions of orders two and three can be calculated by using the formulas below.

$$\mathrm{ber}_2 x = \frac{\sqrt{2}}{x}(-\,\mathrm{ber}_1 x + \mathrm{bei}_1 x) - \mathrm{ber}\,x$$

$$\mathrm{bei}_2 x = \frac{\sqrt{2}}{x}(-\,\mathrm{ber}_1 x - \mathrm{bei}_1 x) - \mathrm{bei}\,x$$

$$\mathrm{ber}_3 x = \frac{2\sqrt{2}}{x}(-\,\mathrm{ber}_2 x + \mathrm{bei}_2 x) - \mathrm{ber}_1 x$$

$$= -\frac{8}{x^2}\mathrm{bei}_1 x - \mathrm{ber}_1 x + \frac{2\sqrt{2}}{x}(\mathrm{ber}\,x - \mathrm{bei}\,x)$$

$$\mathrm{bei}_3 x = \frac{2\sqrt{2}}{x}(-\,\mathrm{ber}_2 x - \mathrm{bei}_2 x) - \mathrm{bei}_1 x$$

$$= \frac{8}{x^2}\mathrm{ber}_1 x - \mathrm{bei}_1 x + \frac{2\sqrt{2}}{x}(\mathrm{ber}\,x + \mathrm{bei}\,x)$$

$$\mathrm{ker}_2 x = \frac{\sqrt{2}}{x}(-\,\mathrm{ker}_1 x + \mathrm{kei}_1 x) - \mathrm{ker}\,x$$

$$\mathrm{kei}_2 x = \frac{\sqrt{2}}{x}(-\,\mathrm{ker}_1 x - \mathrm{kei}_1 x) - \mathrm{kei}\,x$$

$$\mathrm{ker}_3 x = \frac{3\sqrt{2}}{x}(-\,\mathrm{ker}_2 x + \mathrm{kei}_2 x) - \mathrm{ker}_1 x$$

$$= -\frac{8}{x^2}\mathrm{kei}_1 x - \mathrm{ker}_1 x + \frac{2\sqrt{2}}{x}(\mathrm{ker}\,x - \mathrm{kei}\,x)$$

$$\mathrm{kei}_3 x = \frac{2\sqrt{2}}{x}(-\,\mathrm{ker}_2 x - \mathrm{kei}_2 x) - \mathrm{kei}_1 x$$

$$= \frac{8}{x^2}\mathrm{ker}_1 x - \mathrm{kei}_1 x + \frac{2\sqrt{2}}{x}(\mathrm{ker}\,x + \mathrm{kei}\,x)$$

For $x > 10$ (if the Tölke tables are not available, or for $x > 21$ if they are), $\mathrm{ber}_2 x$, $\mathrm{bei}_2 x$, $\mathrm{ker}_2 x$, $\mathrm{kei}_2 x$ and their first derivatives can be calculated with reasonable effort by means of the asymptotic expansions which appear as formulas 185-188, 230-233, pp. 169-172, of reference (7).

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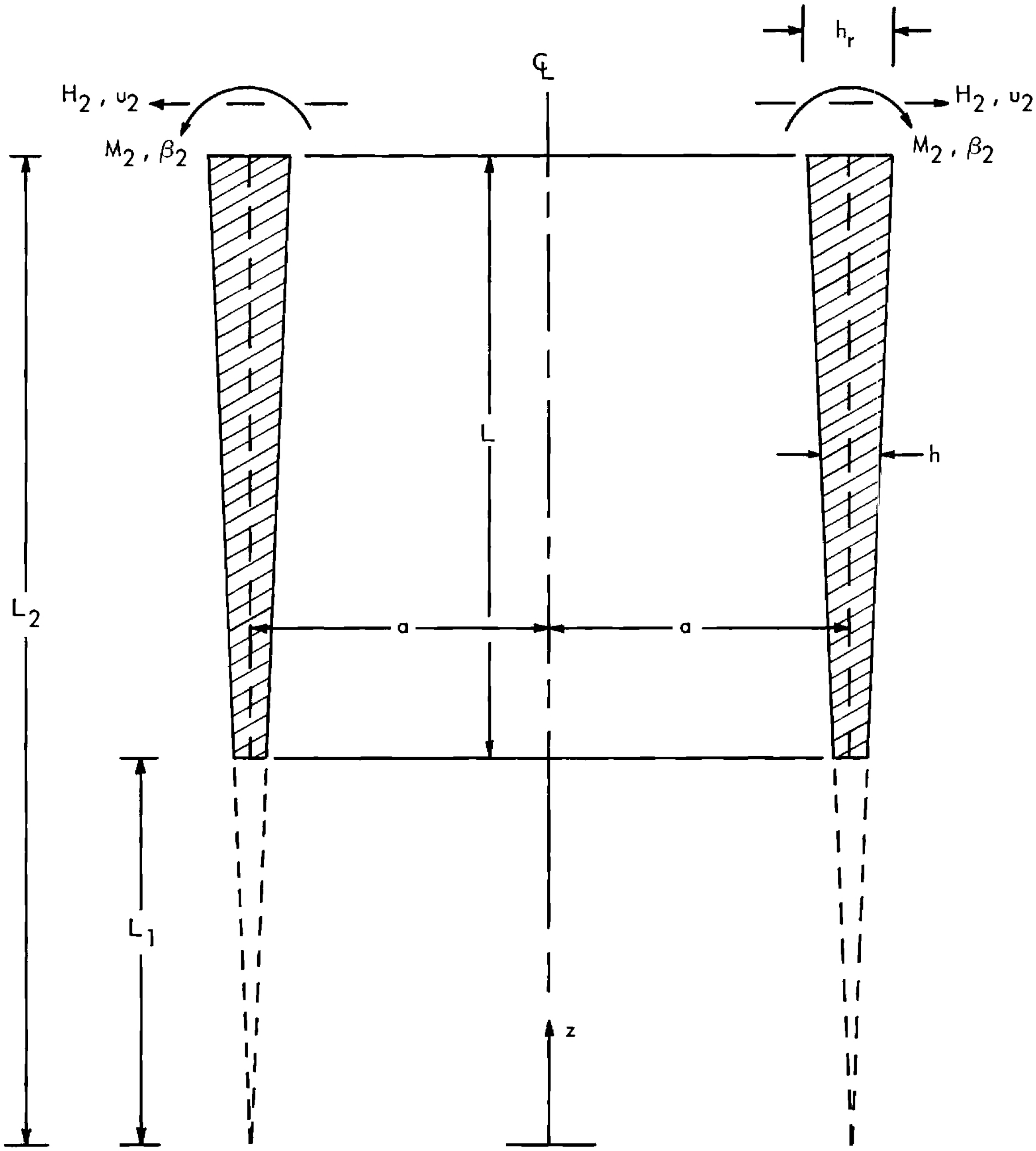


Figure 1. Cross-Section of Shell Loaded Along One Edge.

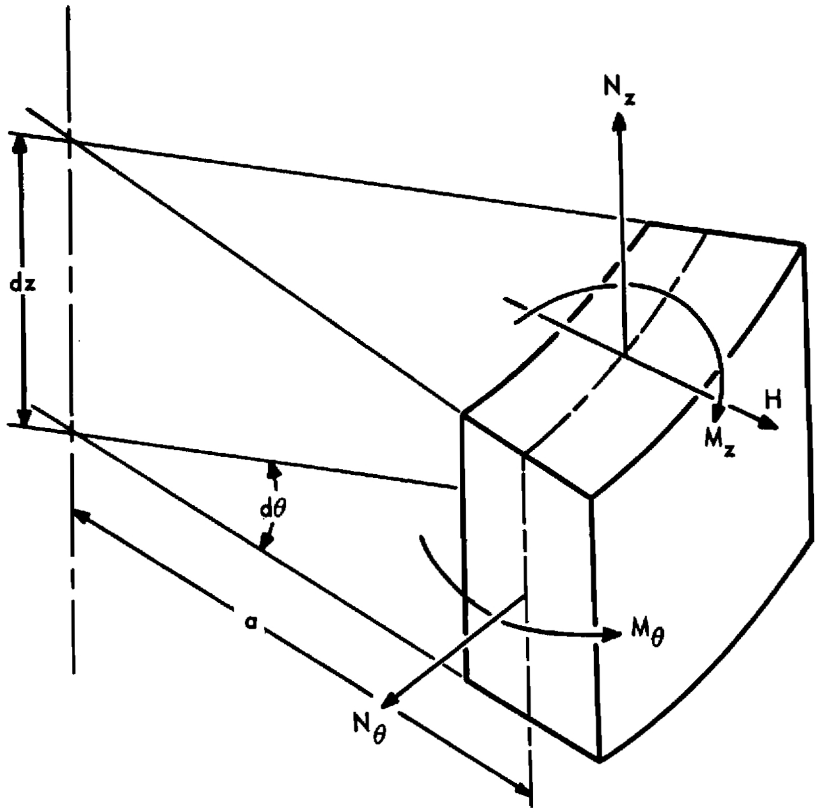


Figure 2. Stress resultants and stress couples acting on an element of the shell.

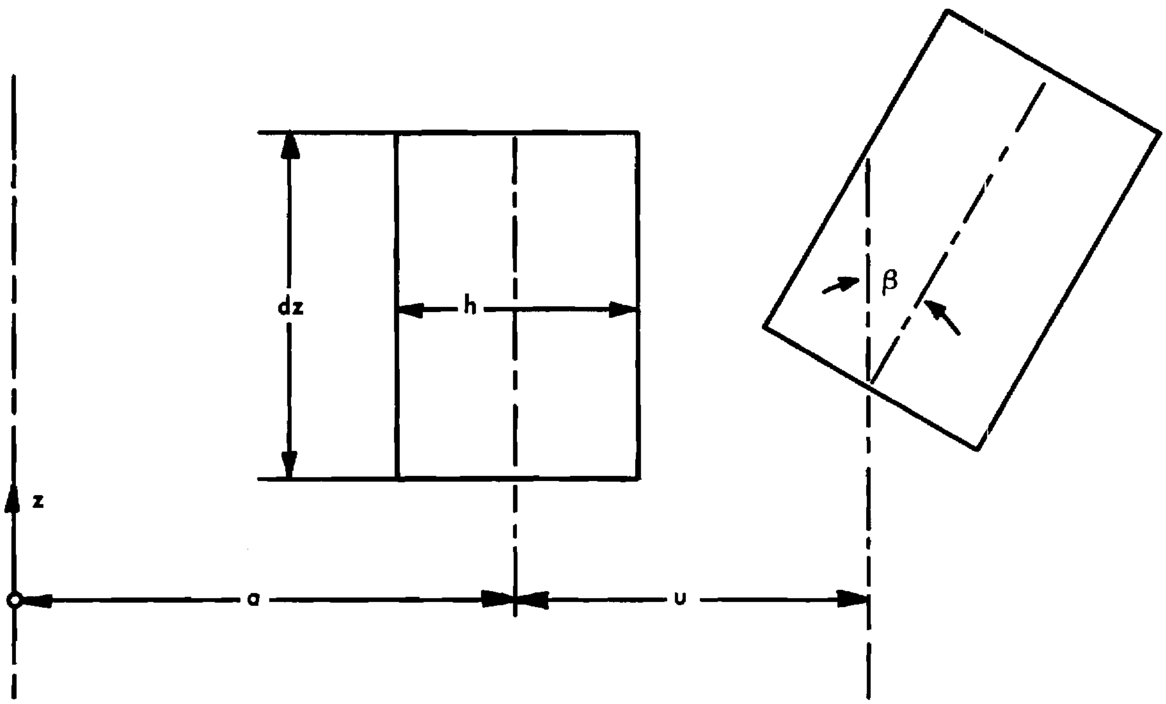
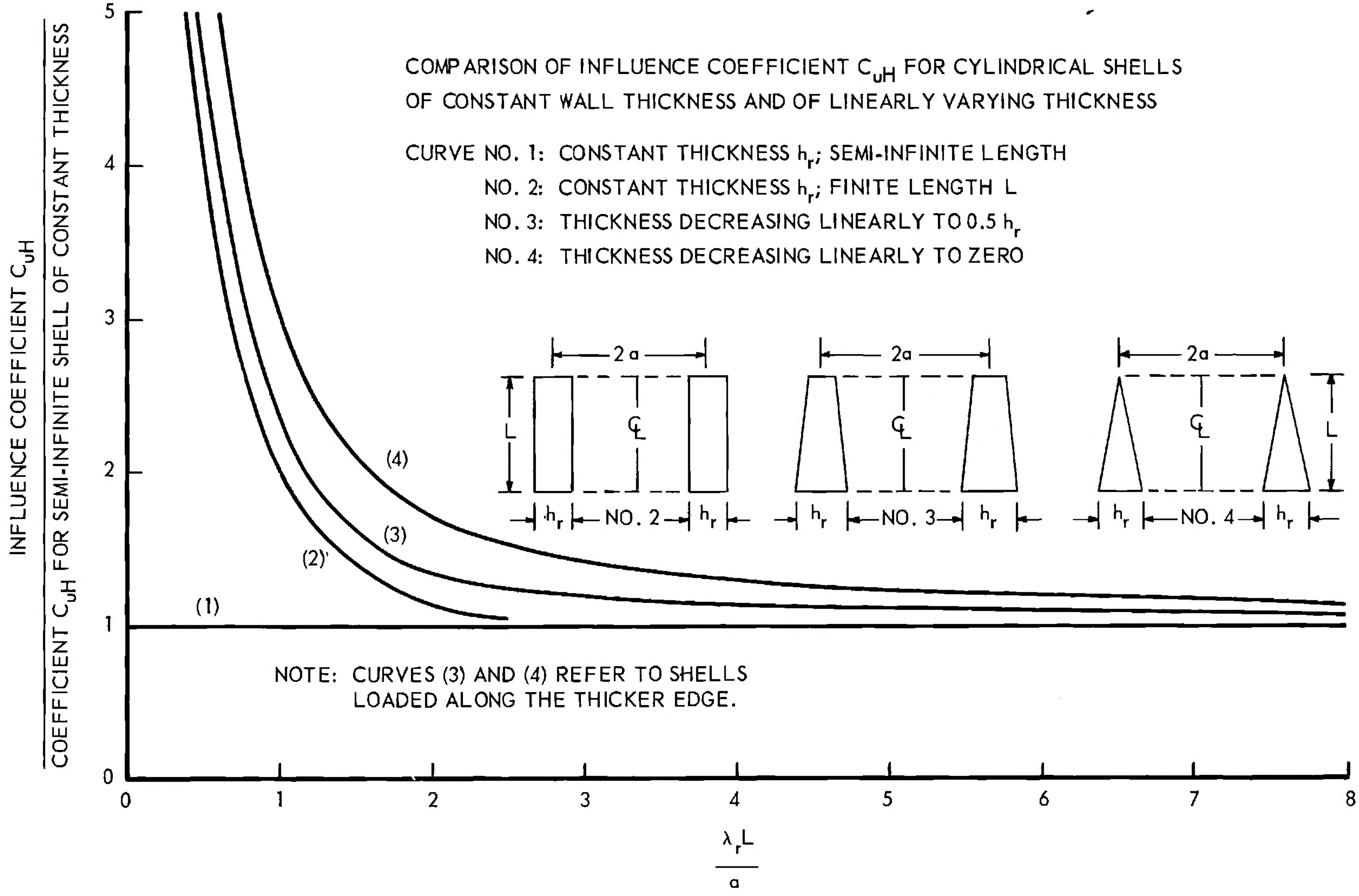


Figure 3. Radial displacement u of a point on the middle surface and rotation β of a meridian through that point.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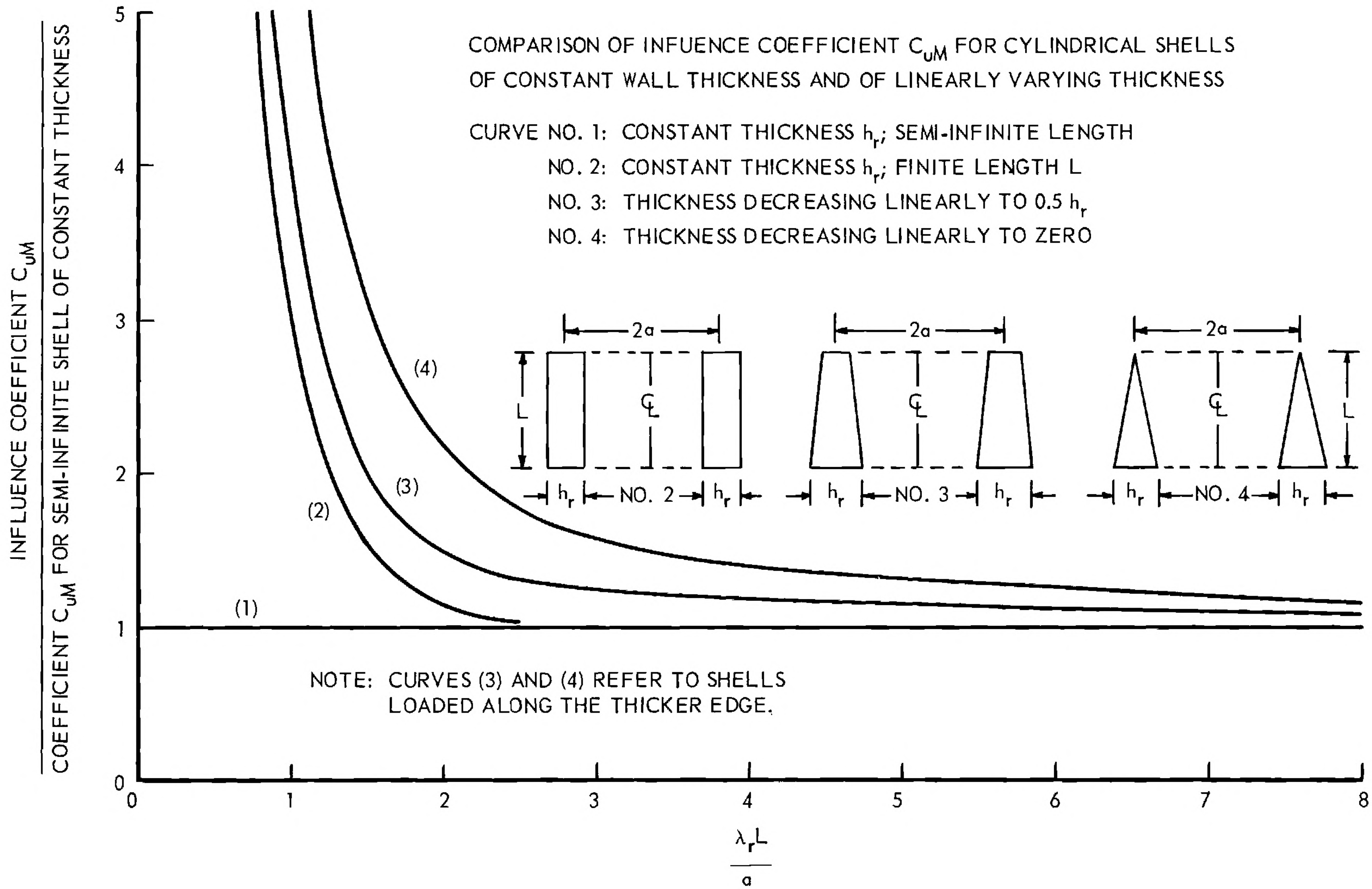
CURVE NO. 1: CONSTANT THICKNESS h_r; SEMI-INFINITE LENGTH
NO. 2: CONSTANT THICKNESS h_r; FINITE LENGTH L
NO. 3: THICKNESS DECREASING LINEARLY TO 0.5 h_r
NO. 4: THICKNESS DECREASING LINEARLY TO ZERO
NOTE: CURVES (3) AND (4) REFER TO SHELLS
LOADED ALONG THE THICKER EDGE.
INFLUENCE COEFFICIENT C_{uH}
COEFFICIENT C_{uH} FOR SEMI-INFINITE SHELL OF CONSTANT THICKNESS
$\frac{\lambda_r L}{a}$
(1)
(2)
(3)
(4)
2a
L
h_r
NO. 2
NO. 3
NO. 4
0
1
2
3
4
5
6
7
8

Figure 4.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NO. 2: CONSTANT THICKNESS h_r; FINITE LENGTH L
NO. 3: THICKNESS DECREASING LINEARLY TO 0.5 h_r
NO. 4: THICKNESS DECREASING LINEARLY TO ZERO
NOTE: CURVES (3) AND (4) REFER TO SHELLS
LOADED ALONG THE THICKER EDGE.
INFLUENCE COEFFICIENT C_{uM} / COEFFICIENT C_{uM} FOR SEMI-INFINITE SHELL OF CONSTANT THICKNESS
$\frac{\lambda_r L}{a}$
(1)
(2)
(3)
(4)
2a
L
h_r
NO. 2
NO. 3
NO. 4
0
1
2
3
4
5
6
7
8

Figure 5.

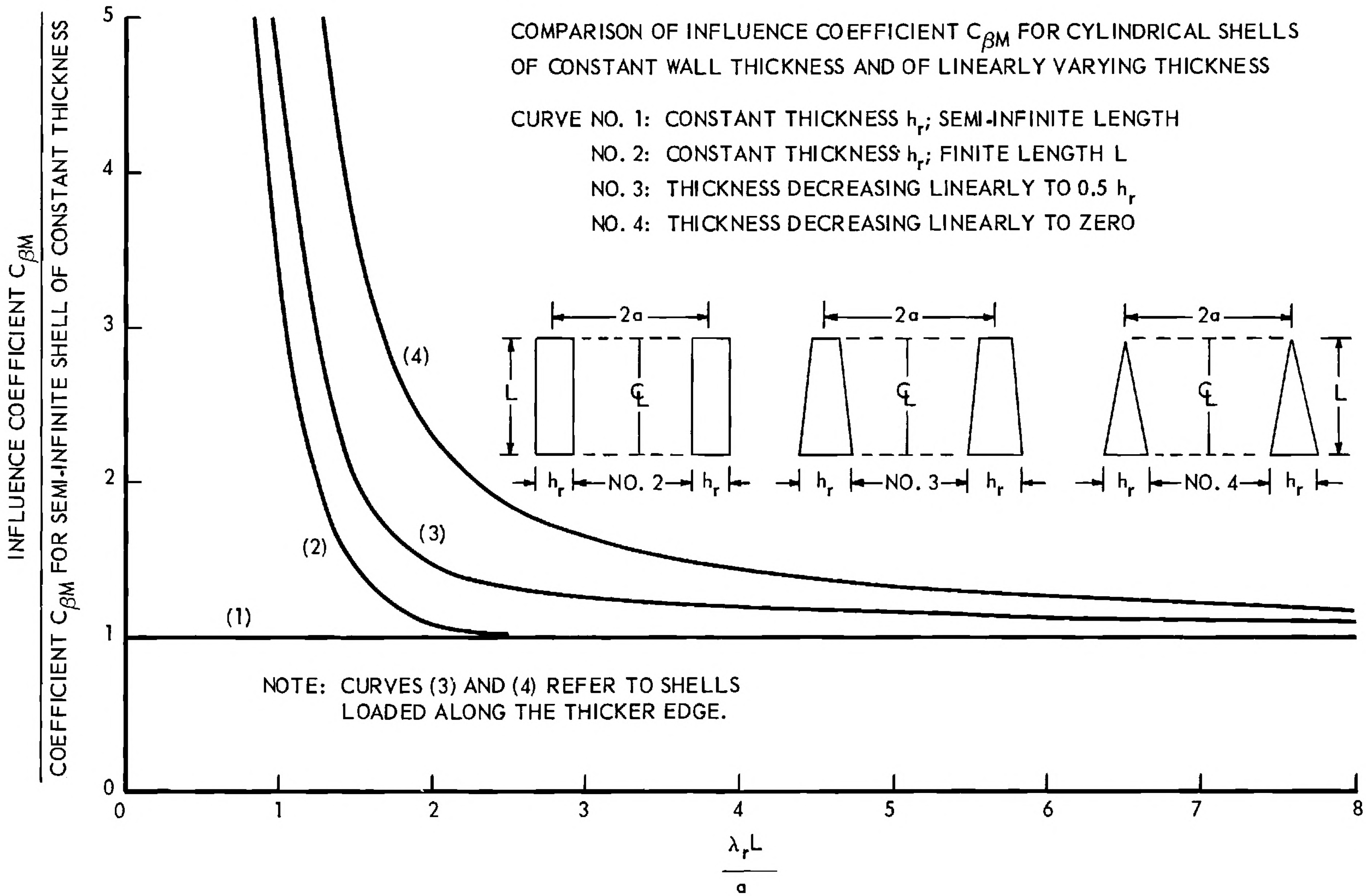
COMPARISON OF INFLUENCE COEFFICIENT $C_{\beta M}$ FOR CYLINDRICAL SHELLS
OF CONSTANT WALL THICKNESS AND OF LINEARLY VARYING THICKNESS
CURVE NO. 1: CONSTANT THICKNESS h_r; SEMI-INFINITE LENGTH
NO. 2: CONSTANT THICKNESS h_r; FINITE LENGTH L
NO. 3: THICKNESS DECREASING LINEARLY TO 0.5 h_r
NO. 4: THICKNESS DECREASING LINEARLY TO ZERO
2a
L
h_r
NO. 2
NO. 3
NO. 4
(1)
(2)
(3)
(4)
NOTE: CURVES (3) AND (4) REFER TO SHELLS LOADED ALONG THE THICKER EDGE.
INFLUENCE COEFFICIENT $C_{\beta M}$ / COEFFICIENT $C_{\beta M}$ FOR SEMI-INFINITE SHELL OF CONSTANT THICKNESS
$\frac{\lambda_r L}{a}$
0
1
2
3
4
5
6
7
8

Figure 6.

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